Radiation of linear waves in the stationary flow of a Bose-Einstein condensate past an obstacle

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Using stationary solutions of the linearized two-dimensional Gross-Pitaevskii equation, we describe the wave pattern occurring in the supersonic flow of a Bose-Einstein condensate past an obstacle. It is shown that these waves are generated outside the Mach cone. The developed analytical theory is confirmed by numerical simulations of the flow past body problem in the frame of the full nonstationary Gross-Pitaevskii equation. Relation of the developed theory with recent experiments is discussed.

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I. INTRODUCTION

Experimental creation of Bose-Einstein condensate (BEC) has led to emergence of a new field of nonlinear wave dynamics owing to a remarkable richness of nonlinear wave patterns supported by this medium. First, vortices and bright and dark solitons were observed and their dynamics were studied theoretically in framework of the mean-field approach (see, e.g., Ref. [1], and references therein). Then, dispersive shocks generated by a large and sharp disturbance of BEC were found in experiment [2,3] and explained theoretically [3,4] in the framework of the Whitham theory of modulations of nonlinear waves (see also the numerical experiment in Ref. [5]). At last, the stationary waves generated by supersonic flow of BEC past obstacles have recently been observed [6]. They were studied in Refs. [7–9] where the main focus was on the nonlinear component representing a train of solitons (a single soliton in the simplest case, see [8]) or, more precisely, having a form of a modulated nonlinear periodic wave. The theory developed in Refs. [7–9] shows that there exist stationary spatial solutions of the Gross-Pitaevskii (GP) equation which describe nonlinear waves supported by a supersonic BEC flow with Mach number

$$M = \frac{u}{c_s} > 1, \tag{1}$$

where u is velocity of the oncoming flow at $x \to -\infty$ and c_s is the sound speed of the long-wavelength linear waves. The density n of the condensate, as well as the components of the velocity field, depend on the variable

$$w = x - ay \tag{2}$$

alone, where a is the slope of the phase lines with respect to the y axis and it is supposed that the velocity of the oncoming flow is directed along the x axis. Then, the Mach cone for sound waves with infinitely large wavelength corresponds to the slope

$$a_M = \sqrt{M^2 - 1}, \tag{3}$$

and it was shown in Ref. [9] that the spatial (oblique) solitons have $a > a_M$, that is they are located inside the Mach cone. In particular, it was shown that the shallow solitons are formed close to the Mach cone $a - a_M \ll a_M$ and are asymptotically described by the Korteweg–de Vries (KdV) equation; and deep solitons have $a \gg 1$ (i.e., they are formed at small angles with respect to direction of the oncoming flow) and are asymptotically described by the nonlinear Schrödinger (NLS) equation.

On the contrary, the linear waves are generated outside the Mach cone and they have $a < a_M$. In fact, these linear waves had been observed in numerical simulations some years ago [10] but a complete theory of their generation has not been developed so far. However, this task is quite topical in view of recent experiments [6]. In connection with this experiment, in the recent paper [11] the linear waves generated by the flow of condensate past an obstacle were interpreted as a Cherenkov radiation of Bogoliubov excitations, some properties of the emerging wave pattern were derived, and nonstationary numerical simulation was performed which showed good agreement with the experiment. The aim of this paper is to develop an analytical theory of the stationary wave pattern and to compare it with the numerical simulations. Although we consider the wave pattern generated by the BEC flow, one should notice that the method used here can be applied to description of other similar effects, in particular, to waves generated by polaritons flow past a defect in a semiconductor microcavity which were discussed in Refs. [12,13].

II. LINEAR WAVES GENERATED IN A BEC FLOW PAST AN OBSTACLE

Our analysis is based on the use of the mean-field description of BEC dynamics in the framework of the GP equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U(\mathbf{r})\psi + g|\psi|^2\psi, \tag{4}$$

where $\psi(\mathbf{r})$ is the order parameter ("condensate wave function"), $U(\mathbf{r})$ is the potential which confines atoms of a Bose gas in a trap and/or describes interaction of the BEC with the obstacle, and g is an effective coupling constant arising due to interatomic collisions with the s-wave scattering length a_s,

$$g = 4\pi\hbar^2 a_s/m, \tag{5}$$

m being the atomic mass. Here we consider the Bose-Einstein condensate with repulsive interaction between particles for which $g > 0$.

As suggested by the experimental setup [6], we consider a two-dimensional flow of a condensate, so that the condensate wave function ψ depends on only two spatial coordinates, $\mathbf{r} = (x, y)$. To simplify the theory, we assume that the characteristic size of the obstacle is much less than its distance from the center of the trap, so that the oncoming flow can be considered as uniform with constant density n_0 of atoms and constant velocity $\mathbf{u}_0$ directed parallel to x axis (see also estimates in Ref. [8]). It is convenient to transform Eq. (4) to a "hydrodynamic" form by means of the substitution

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp\left(\frac{i}{\hbar}\int^{\mathbf{r}} \mathbf{u}(\mathbf{r}',t)d\mathbf{r}'\right), \tag{6}$$

where $n(\mathbf{r},t)$ is density of atoms in BEC and $\mathbf{u}(\mathbf{r},t)$ denotes its velocity field, and to introduce dimensionless variables

$$\tilde{x} = x/\sqrt{2}\xi, \quad \tilde{y} = y/\sqrt{2}\xi, \quad \tilde{t} = (c_s/2\sqrt{2}\xi)t,$$

$$\tilde{n} = n/n_0, \quad \tilde{\mathbf{u}} = \mathbf{u}/c_s, \tag{7}$$

where $\xi = \hbar/\sqrt{2mn_0 g}$ is the BEC healing length and numerical constants are introduced for future convenience. As a result of this transformation we obtain the system (we omit tildes for convenience of the notation)

$$\frac{1}{2}n_t + \boldsymbol{\nabla}(n\mathbf{u}) = 0,$$

$$\frac{1}{2}\mathbf{u}_t + (\mathbf{u}\,\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}n + \boldsymbol{\nabla}\left[\frac{(\boldsymbol{\nabla}n)^2}{8n^2} - \frac{\Delta n}{4n}\right] = 0, \tag{8}$$

where $\boldsymbol{\nabla} = (\partial_x, \partial_y)$. Since we shall consider waves far enough from the obstacle, the potential is omitted in Eq. (8).

We are interested in linear waves propagating on the background flow with $n=1$, $u=M$, $v=0$. Hence, we introduce

$$n = 1 + n_1, \quad u = M + u_1, \quad v = v_1, \tag{9}$$

and linearize the system (8) with respect to small deviations n_1, u_1, v_1. As a result we obtain the system

$$\frac{1}{2}n_{1,t} + u_{1,x} + Mn_{1,x} + v_{1,y} = 0,$$

$$\frac{1}{2}u_{1,t} + Mu_{1,x} + n_{1,x} - \frac{1}{4}(n_{1,xxx} + n_{1,xyy}) = 0,$$

$$\frac{1}{2}v_{1,t} + Mv_{1,x} + n_{1,y} - \frac{1}{4}(n_{1,xxy} + n_{1,yyy}) = 0, \tag{10}$$

which describes propagation of linear waves in BEC with a uniform flow. We obtain the applicability condition of these equations, if we notice that in the linear wave $u_1 \sim Mn_1$ and the nonlinear terms of the order of magnitude $u_1 \boldsymbol{\nabla} u_1 \sim \boldsymbol{\nabla}(Mn_1)^2$ can be neglected as long as they are much less than the linear ones $\sim \boldsymbol{\nabla} n_1$. Thus, we get the criterion

$$n_1 \ll 1/M^2. \tag{11}$$

Hence, if M is large enough, the linear theory is applicable to description of waves outside the Mach cone far enough from the obstacle, where the density amplitude n_1 of the wave satisfies the condition (11). For harmonic waves $n_1, u_1, v_1 \propto \exp[i(k_x x + k_y y) - i\omega t]$ the system (10) yields at once the dispersion relation

$$\frac{\omega}{2} = Mk_x \pm k\sqrt{1 + \frac{k^2}{4}}, \tag{12}$$

where $k = \sqrt{k_x^2 + k_y^2}$. Actually, this is a well-known dispersion relation of the Bogoliubov excitations in BEC with a flow (see, e.g., Ref. [11]).

Now we consider the stationary wave patterns far enough from the obstacle where the condition (11) is supposed to be fulfilled. In fact, this problem is analogous to the Kelvin theory of ship waves generated by a ship moving in a deep water, but with a different dispersion law (12). In this method, the wave pattern stationary in the ship reference system is represented as a wave packet propagating with the group velocity equal to the ship velocity. Hence, it can be considered in framework of the modulation theory of linear waves. Here we shall follow Kelvin's method in its form presented in Refs. [14,15].

First, we notice that in a stationary wave pattern $\omega = 0$ and, hence, the components of the wave vector $\mathbf{k} = (k_x, k_y)$ are the functions of the space coordinates (x, y) connected with each other by the relationship

$$G(k_x, k_y) \equiv Mk_x + k\sqrt{1 + \frac{k^2}{4}} = 0, \tag{13}$$

where we have taken into account that for chosen geometry of the BEC flow the wave must propagate upwind, i.e., $k_x < 0$.

Next, the "ship wave" pattern corresponds to a modulated two-dimensional wave where the wave vector $\mathbf{k}$ is a gradient of the phase [14],

$$\theta = \int_0^{\mathbf{r}} \mathbf{k} \cdot d\mathbf{r}. \tag{14}$$

Hence, the components (k_x, k_y) satisfy the condition

$$\frac{\partial k_x}{\partial y} - \frac{\partial k_y}{\partial x} = 0, \tag{15}$$

which, with an account of Eq. (13), yields the equation for k_y

$$\frac{\partial k_y}{\partial x} - f'(k_y)\frac{\partial k_y}{\partial y} = 0, \tag{16}$$

where $f'(k_y)$ is defined by the derivative of an implicit function (13):

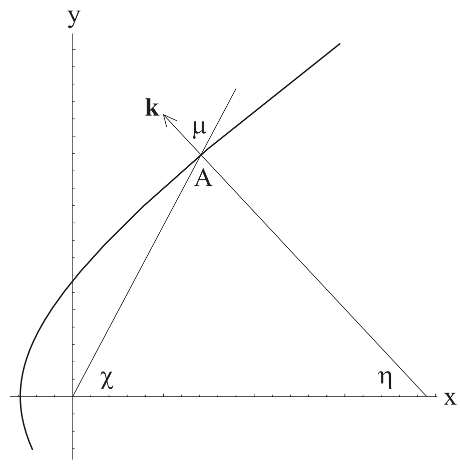

FIG. 1. Sketch of the wave front with wave vector **k** normal to it at the point A and coordinates χ and η defining the parameters of the wave pattern at this point.

$$f' = -\frac{\partial G/\partial k_y}{\partial G/\partial k_x}. \tag{17}$$

It follows from Eq. (16) that k_x and k_y are constant along the characteristics defined as solutions of the equation

$$\frac{dy}{dx} = -f'(k_y). \tag{18}$$

At last, since at large distances from the obstacle, the latter can be considered as a pointlike source of waves, the resulting stationary wave is centered, that is the characteristics intersect at the point $(x,y)=(0,0)$ of the location of the obstacle. Hence, we obtain the solution

$$\frac{y}{x} = \tan\chi = -\frac{\partial G/\partial k_y}{\partial G/\partial k_x}, \tag{19}$$

where χ is the angle between the x axis and the radius vector $\mathbf{r}$ of the point A with wave vector (k_x,k_y); see Fig. 1 where other convenient parameters are also defined, namely, the angle η between the wave vector $\mathbf{k}$ and a negative direction of the x axis, and the angle $\mu=\pi-\chi-\eta$ between vectors $\mathbf{k}$ and $\mathbf{r}$. Since the function $G(k_x,k_y)$ is the dispersion law for linear (Bogoliubov) waves, the relation (19) can be obviously interpreted as follows: the ratio of the y and x coordinates in the wave pattern formed by a pointlike wave source is equal to the ratio of the corresponding components of the group velocity vector (see Ref. [11]).

Thus, we have

$$k_x = -k\cos\eta, \quad k_y = k\sin\eta, \tag{20}$$

$$G(k,\eta) = -Mk\cos\eta + k\sqrt{1+\frac{k^2}{4}} = 0. \tag{21}$$

Then elementary calculation gives for Eq. (19) the expression

$$\tan\chi = \frac{(2M^2-1-\tan^2\eta)\tan\eta}{(M^2+1)\tan^2\eta-(M^2-1)} \tag{22}$$

and Eq. (21) yields

$$k = 2\sqrt{M^2\cos^2\eta-1}. \tag{23}$$

Thus, we have found that for a fixed value of η the components (k_x,k_y) are constant along the line χ=const with χ defined by Eq. (22) and the length of the wavevector given by Eq. (23). Therefore the phase (14) can be conveniently calculated by integration along the line χ=const with constant vector $\mathbf{k}$, so that

$$\theta = (k\cos\mu)r. \tag{24}$$

This means that the lines of constant phase (e.g., the wave crests) θ are determined in parametrical form by Eq. (22) and

$$r = \frac{\theta}{k\cos\mu}, \tag{25}$$

where k is given by Eq. (23) and μ can be calculated from $\tan\mu=-\tan(\chi+\eta)$ which gives, after elementary algebra, the expression

$$\tan\mu = \frac{2M^2}{k^2}\sin 2\eta. \tag{26}$$

This expression permits one to express Eq. (25) as

$$r = \frac{4\theta}{k^3}\sqrt{M^2(M^2-2)\cos^2\eta+1} \tag{27}$$

and Eq. (22) as

$$\tan\chi = \frac{(1+k^2/2)\tan\eta}{M^2-(1+k^2/2)}. \tag{28}$$

At last, the curves with constant phase θ are given in Cartesian coordinates by the formulas

$$x = r\cos\chi = \frac{4\theta}{k^3}\cos\eta(1-M^2\cos 2\eta),$$

$$y = r\sin\chi = \frac{4\theta}{k^3}\sin\eta(2M^2\cos^2\eta-1). \tag{29}$$

Thus, we have found the expressions describing the linear wave pattern in a parametric form where the parameter η changes in the interval

$$-\arccos\frac{1}{M} \leqslant \eta \leqslant \arccos\frac{1}{M}. \tag{30}$$

For $\eta=0$ we have $x<0$ and $y=0$, that is small values of η correspond to the wave before the obstacle. Series expansions of Eqs. (29) give for small η

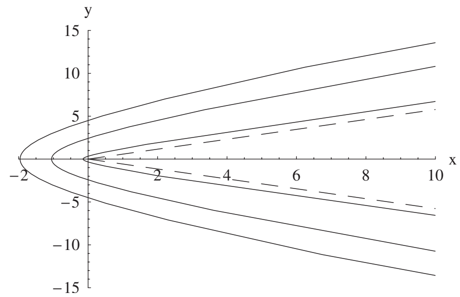

FIG. 2. Wave pattern of stationary linear waves generated in the flow of BEC past a pointlike obstacle. Dashed line corresponds to the Mach cone of linear waves in the long wavelength limit.

$$x \cong -\frac{\theta}{2\sqrt{M^2-1}} + \frac{(2M^2-1)\theta}{4(M^2-1)^{3/2}}\eta^2,$$

$$y \cong \frac{(2M^2-1)\theta}{2(M^2-1)^{3/2}}\eta, \tag{31}$$

hence the wave crests have here a parabolic form

$$x(y) \cong -\frac{\theta}{2\sqrt{M^2-1}} + \frac{(M^2-1)^{3/2}}{(2M^2-1)\theta}y^2. \tag{32}$$

The boundary values $\eta = \pm \arccos(1/M)$ correspond to the lines

$$\frac{x}{y} = \pm\sqrt{M^2-1} = \pm a_M, \tag{33}$$

that is, the curves of constant phase become asymptotically the straight lines parallel to the Mach cone. The general pattern is shown in Fig. 2.

III. NUMERICAL SIMULATIONS

We have compared the above approximate analytical theory with the exact numerical solution of the GP equation, which in nondimensional units (7) takes the standard form

$$i\psi_t = -\frac{1}{2}(\psi_{xx} + \psi_{yy}) + U(x,y)\psi + |\psi|^2\psi, \tag{34}$$

which corresponds for $U=0$ to the system (8) with

$$\psi = \sqrt{n}\exp\left(i\int^{\mathbf{r}} \mathbf{u}(\mathbf{r}',t)d\mathbf{r}'\right).$$

In our simulations the obstacle was modeled by an impenetrable disk with radius $r=1$. Such an obstacle introduces large enough perturbation into the flow to generate an oblique solitons pair behind it (see Ref. [8]). We assume that at the initial moment $t=0$ there is no disturbance in the condensate, so that it is described by the plane wave function

$$\psi(x,y)|_{t=0} = \exp(iMx)$$

corresponding to a uniform condensate flow. For large enough evolution time, the wave pattern around the obstacle tends to a stationary structure. An example of such a structure for $M=2$ is shown in Fig. 3. A dashed line corresponds to the analytic theory developed in the preceding section; as we see it agrees very well with the numerical results for $M=2$.

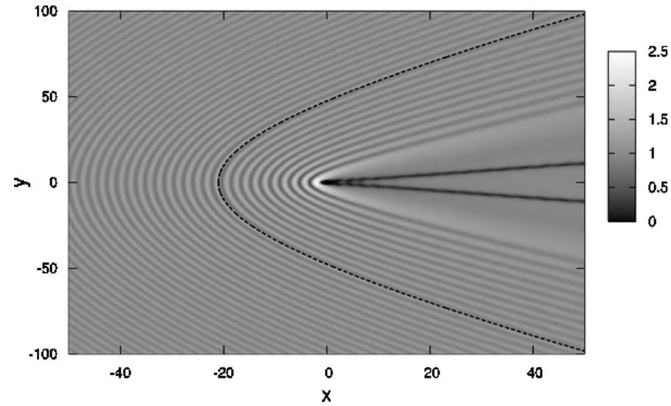

FIG. 3. Numerically calculated wave pattern of stationary linear waves generated in the flow of BEC past an obstacle. The Mach number is equal to $M=2$ and the radius of an impenetrable obstacle to $r=1$. Dashed line corresponds to the linear analytical theory for the line of constant phase. It corresponds to Eqs. (29) but the curve is shifted to 1 unit of length to the left from the center of the obstacle for better fitting to numerics. It should be noted that this distance is negligibly small in our theory corresponding to a pointlike obstacle. A pair of oblique dark lines behind the obstacle correspond to spatial solitons studied in Ref. [8].

The condition (11) indicates that the nonlinear effects grow up with increase of M. To demonstrate this explicitly, we have compared the wavelength λ at $y=0$ calculated using the developed linear analytic theory with the same parameter obtained from our full numerical simulations. According to linear theory, the wavelength at $y=0$ (i.e., $\eta=0$) is constant and equal to

$$\lambda = \frac{2\pi}{k} = \frac{\pi}{\sqrt{M^2-1}}, \tag{35}$$

where we have used Eq. (23) (similar expression was derived in Ref. [11]). In Fig. 4 we compare this dependence of the wavelength λ on the Mach number M with the results of numerical simulations at the point with $n_1 \approx 0.1$. As we see, Eq. (35) is very accurate for values of M satisfying the condition (11) and discrepancy between analytical and numerical results slightly increases with increase of M. In general, this plot confirms validity of a linear theory in the region of its applicability.

Similar numerical simulations have been performed in a recent paper [11]. However, in [11] the case of time-dependent flow changing with time Mach number and other parameters was discussed. Correspondingly, the stationary pattern described by the present theory has not been achieved. For comparison of the present theory with the experiment [6] one should take into account that in the experiment the obstacle was located very close to the center of the trap and it was large enough to create a shadow behind it. Therefore it would take quite a long time to reach the quasi-stationary stage of BEC evolution with considerable density

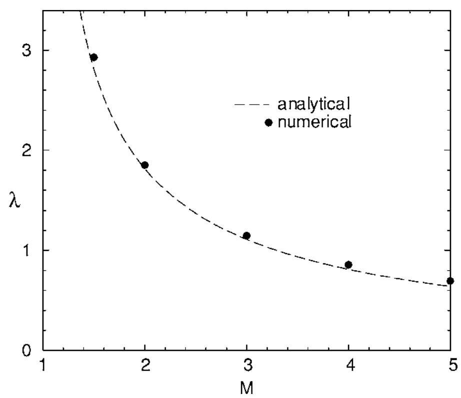

FIG. 4. Dependence of the wavelength of the wave generated before the obstacle on the Mach number calculated numerically (full time-dependent GP equation)—dots, and analytically (linear theory)—dashed line.

of the condensate behind the obstacle. We suppose that for this reason the oblique solitons predicted in [8] were not visible in this experiment. Nevertheless, the general picture of arising experimental pattern of linear waves agrees qualitatively with the pattern described by the present analytical theory and observed in numerical simulations.

IV. CONCLUSION

We have developed here the theory of linear waves generated by the flow of BEC past an obstacle. The linear approximation is correct for small enough amplitudes of the perturbation. This condition is satisfied in the case of small disturbance introduced by the obstacle and not too high values of the Mach number. Our numerical simulations confirm the analytical theory in the region of its applicability.

The described here wave pattern corresponds to a stationary limit of patterns observed in recent experiments [6] (see also Ref. [11]). As was shown in Ref. [8], at long enough time of free expansion of the condensate released form a trap, there exists a region around the obstacle where the flow can be considered as almost uniform and quasistationary. The developed here theory can be applied to the quantitative description of the experimental data corresponding to this region of the BEC flow.

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