Quantum Landau damping in dipolar Bose-Einstein condensates

J. T. Mendonça,[1,*] H. Terças,[1,†] and A. Gammal[2,‡]
[1]*IPFN, Instituto Superior Técnico, Universidade de Lisboa, Lisboa 1049-001, Portugal*
[2]*Instituto de Física, Universidade de São Paulo, São Paulo 05508-090, Brazil*

(Received 19 January 2018; published 12 June 2018)

We consider Landau damping of elementary excitations in Bose-Einstein condensates (BECs) with dipolar interactions. We discuss quantum and quasiclassical regimes of Landau damping. We use a generalized wave-kinetic description of BECs which, apart from the long-range dipolar interactions, also takes into account the quantum fluctuations and the finite-energy corrections to short-range interactions. Such a description is therefore more general than the usual mean-field approximation. The present wave-kinetic approach is well suited for the study of kinetic effects in BECs, such as those associated with Landau damping, atom trapping, and turbulent diffusion. The inclusion of quantum fluctuations and energy corrections changes the dispersion relation and the damping rates, leading to possible experimental signatures of these effects. Quantum Landau damping is described with generality, and particular examples of dipolar condensates in two and three dimensions are studied. The occurrence of roton-maxon excitations, and their relevance to Landau damping, are also considered in detail. The present approach is mainly based on a linear perturbative procedure, but the nonlinear regime of Landau damping, which includes atom trapping and atom diffusion, is also briefly discussed.

DOI: 10.1103/PhysRevA.97.063610

I. INTRODUCTION

Many-body effects in quantum states of matter form probably one of the most challenging topics of research in modern physics. In this context, the phenomenon of superfluidity is a prominent example, fascinating and thrilling scientists for almost a century [1,2]. An enormous advance in the understanding of superfluidity was due to Landau, who introduced the concept of quasiparticles [3]. In order to explain the special thermodynamic features of ^{4}He, Landau postulated the existence of two types of excitations: phonons, the long-wavelength acoustic waves, and rotons, the gapped excitations taking place at finite wave vectors k. Initially understood as localized vortices, rotons have been observed in neutron-scattering experiments and are now understood to be related to the correlated nature of the interactions in liquid helium [4].

Another physical system featuring roton excitations is that of dipolar gases [5]. Dipolar Bose-Einstein condensates (BECs) differ from their short-range interacting counterparts [6,7] by the addition of a long-range, anisotropic potential [8]. In some cases, a convenient manipulation of external electric or magnetic fields can nearly remove the short-range potential, and dipolar forces become dominant [9]. Contrary to what happens in liquid helium, where rotons are attributed to strong correlations, low-dimensional dipolar gases feature rotons as a consequence of the long-range character of the dipole-dipole potential [10]. The experimental observation of rotons in dipolar BECs has recently been reported in Ref. [11]. Meanwhile, BECs loaded in an optical cavity also became known for exhibiting roton modes [12,13].

At the mean-field level, dipolar BECs are described by a nonlocal Gross-Pitaevskii (GP) equation, accounting for the long-range, anisotropic dipolar potential [8]. Nevertheless, generalized and more accurate forms of the GP equation have been recently proposed [14,15], displaying cubic and quartic nonlinearities. The usual cubic term describes two-body collisions at zero energy, and the quartic term represents the Lee-Huang-Yang (LHY) correction resulting from quantum fluctuations [16–19]. The LHY correction has been shown to be essential in explaining the appearance of droplets in dipolar condensates [20,21]. Additional cubic terms, resulting from a first-order energy correction to the two-body collisions, have also appeared in the literature [22].

Landau damping, i.e., the damping of the collective excitations via the isentropic transfer of their energy to the particles of the system [23,24], has been investigated in the past for the case of contact interacting BECs [25–28], and its occurrence in dipole BECs has recently been addressed [29]. In the present paper, we extend the previous analysis to consider both the quantum and semiclassical regimes. We also predict additional features, such as atom trapping, quasilinear diffusion, and kinetic instabilities. Our approach is based on the wave-kinetic (WK) description of the system. This alternative approach is particularly powerful to describe kinetic processes (i.e., wave-particle and wave-wave interactions), such as those taking place in the mechanism of Landau damping. The WK description of dipolar BECs is based on the Wigner function associated with the macroscopic wave function of the system, and follows by employing the Wigner-Moyal procedure [30,31], finding application in short-range BECs at finite temperature [32]. In this paper, we use a generalized WK equation, allowing us to establish a direct relation between the

*titomend@tecnico.ulisboa.pt
†hugo.tercas@tecnico.ulisboa.pt
‡gammal@if.usp.br

initial equilibrium distribution of the atoms in a dipolar BEC and the corresponding damping of the collective excitation, without the need to distinguish the initial and the final states of the decay, as is the case of the calculations based on Fermi's "golden rule" [29]. Moreover, our formalism automatically accommodates the two-body collisions and LHY corrections, making it particularly appealing for future applications with quantum droplets [11].

This paper is organized as follows. In Sec. II, we present the WK equation for dipolar BECs, which is the basic equation of this paper. In Sec. III, with the aim of benchmarking our description with respect to more standard techniques used in the cold-atom community, we derive the dispersion relation of elementary excitations. As particular cases, we consider typical three-dimensional (3D) and quasi-two-dimensional (2D) configurations. The 3D case contains unstable regions in the range of large wave numbers, and the quasi-2D case features the roton-maxon dispersion relation [10,11,19]. In Sec. IV, the kinetic Landau damping is considered. We discuss the cases of a finite temperate BEC and show that the dipolar interactions modify the Landau damping rate. The quantum and the quasiclassical regimes are examined. We further discuss the possible occurrence of kinetic two-stream instabilities and their relation with the fluid instability studied in Ref. [33]. We show that Landau damping can still exist for condensates with a finite size, even at zero temperature. Finite dimensions imply the existence of an effective temperature, as a consequence of the velocity broadening associated with the Heisenberg uncertainty principle. This effective temperature is estimated to be quite small when compared to the critical condensation temperature. However, we argue that its effect may become appreciable near the roton minimum, for which the phase velocity approaches zero. In Sec. V, we discuss the limits of validity of the linear Landau damping regime. Such a discussion includes the main processes that could occur in the nonlinear regime, namely, atom trapping and atom diffusion. The former is a consequence of finite amplitude oscillations, and relies on the possible existence of trapped quantum states, while the latter takes place if a broad spectrum of excitations exists. Finally, in Sec. VI, we state some conclusions.

II. WAVE-KINETIC EQUATION

We consider a dipolar condensate, as described by a modified GP equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = (H_{\rm GP} + H')\psi, \tag{1}$$

where $\psi \equiv \psi(\mathbf{r};t)$ is the condensate order parameter and $H_{\rm GP}$ is the usual GP Hamiltonian as determined by

$$H_{\rm GP} = -\frac{\hbar^2}{2m}\nabla^2 + V_0(\mathbf{r}) + g\,|\psi(\mathbf{r},t)|^2. \tag{2}$$

Here, $V_0(\mathbf{r})$ is the confining potential, $g = 4\pi\hbar^2 a/m$ is the strength of the short-range interaction, and a is the scattering length. The Hamiltonian H' in Eq. (1) describes three additional effects and can be written as

$$H' = Q|\psi(\mathbf{r},t)|^3 + \frac{1}{2}\chi[\nabla^2|\psi(\mathbf{r},t)|^2] + \int U_d(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}',t)|^2 d\mathbf{r}'. \tag{3}$$

The first term describes the LHY correction due to quantum fluctuations, determined by the coefficient $Q = g(32/3\sqrt{\pi})a^{3/2}$. The second term is due to the finite-energy range of atom collisions, and the corresponding coefficient is $\chi = g(a - r_e/2)$, with r_e being the effective range obtained from the second-order expansion of the phase shift [14]. Finally, the third term describes the long-range dipolar interactions and is characterized by a dipole interaction potential U_d, to be specified later.

Equation (1) constitutes a proper description of the quantum corrected dynamics of a dipolar condensate. Alternatively, we can describe the condensate considering the autocorrelation function. This can be done invoking the Wigner function, which can be defined as

$$W(\mathbf{q},\mathbf{r},t) = \int \psi^*(\mathbf{r}-\mathbf{s}/2,t)\psi(\mathbf{r}+\mathbf{s}/2,t)\exp(i\mathbf{q}\cdot\mathbf{s})d\mathbf{s}. \tag{4}$$

Starting from the above generalized GP equation, and applying the well-known Wigner-Moyal procedure (see, e.g., [30,32–34] and references therein), we can derive an evolution equation for W, of the form

$$i\hbar\left(\frac{\partial}{\partial t} + \mathbf{v}_q\cdot\nabla\right)W = \int V_k(t)\Delta W\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{k}}{(2\pi)^3}, \tag{5}$$

where $\mathbf{v}_q = \hbar\mathbf{q}/m$ is the atom velocity, and ΔW is defined as

$$\Delta W = W^- - W^+, \quad W^\pm \equiv W(\mathbf{q}\pm\mathbf{k}/2,\mathbf{r},t). \tag{6}$$

The quantity $V_k(t)$ in Eq. (5) is the spatial Fourier transform of the total potential $V(\mathbf{r},t)$. We should notice that the Wigner distribution is normalized to the local atom density as

$$n(\mathbf{r},t) \equiv |\psi(\mathbf{r},t)|^2 = \int W(\mathbf{q},\mathbf{r},t)\frac{d\mathbf{q}}{(2\pi)^3}. \tag{7}$$

This allows us to write the total potential in Eq. (1) as $V(\mathbf{r},t) = [V_0 + gn + Qn^{1/2} + \chi(\nabla^2 n)/2] + V_d(\mathbf{r},t)$, where the dipolar term is determined by

$$V_d(\mathbf{r},t) = \int\frac{d\mathbf{q}}{(2\pi)^3}\int d\mathbf{r}' U_d(\mathbf{r}-\mathbf{r}')W(\mathbf{q},\mathbf{r}',t). \tag{8}$$

From the convolution theorem, we have

$$\int U_d(\mathbf{r}-\mathbf{r}')W(\mathbf{q},\mathbf{r}',t)\,d\mathbf{r}' = \int U_d(\mathbf{k})W_k(\mathbf{q},t)\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d\mathbf{k}}{(2\pi)^3}, \tag{9}$$

where $U_d(\mathbf{k})$ and $W_k(\mathbf{q},t)$ are the Fourier transforms of the dipolar potential $U_d(\mathbf{r})$ and the quasiprobability $W(\mathbf{q},\mathbf{r},t)$, respectively. Plugging into Eq. (8), we can obtain

$$V(\mathbf{r},t) = \int V_k(t)\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d\mathbf{k}}{(2\pi)^3}, \tag{10}$$

with

$$V_k(t) = V_0(\mathbf{k}) + \left[g - \frac{k^2}{2}\chi + U_d(\mathbf{k})\right]n_k(t) + Q\int n^{3/2}(\mathbf{r},t)\exp(-i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}, \tag{11}$$

with $n_k(t)$ being the Fourier transform of Eq. (7). The wave-kinetic equation in Eq. (5), together with the expression for $V_k(t)$ in Eq. (11), provides the full phase-space description of a dipolar BEC in the presence of quantum fluctuations.

III. DISPERSION RELATION

In order to discuss the elementary excitations of the dipolar BEC, we assume that the Wigner function can be divided in two distinct parts, $W = W_0 + \tilde{W}$. Here, W_0 is the equilibrium distribution describing the condensate in steady state, and $\tilde{W}$ is a small perturbation such that $|\tilde{W}| \ll |W_0|$, describing the elementary excitations of the system. For simplicity, we neglect the trap and assume a plane-wave perturbation of the form

$$\tilde{W}(\mathbf{q},\mathbf{r},t) = \tilde{W}_k(\mathbf{q})\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t), \tag{12}$$

where ω is the mode frequency. Linearizing Eq. (5) with respect to the perturbed quantities, we can easily get

$$\tilde{W}_k = \left[g + Q\sqrt{n_0} - \frac{\chi}{2}k^2 + U_d(\mathbf{k})\right]\frac{\Delta W_0}{\hbar(\omega - \mathbf{k}\cdot\mathbf{v}_q)}\tilde{n}_k, \tag{13}$$

where $\Delta W_0 = W_0(\mathbf{q} - \mathbf{k}/2) - W_0(\mathbf{q} + \mathbf{k}/2)$ and $\tilde{n}_k$ is the density fluctuation. Integrating over the momentum distribution, we can then obtain a dispersion relation of the form

$$1 - \left[g + Q\sqrt{n_0} - \frac{\chi}{2}k^2 + U_d(\mathbf{k})\right]\int\frac{\Delta W_0}{\hbar(\omega - \mathbf{k}\cdot\mathbf{v}_q)}\frac{d\mathbf{q}}{(2\pi)^3} = 0, \tag{14}$$

which can explicitly be recast in the form

$$1 - \left[g + Q\sqrt{n_0} - \frac{\chi}{2}k^2 + U_d(\mathbf{k})\right]\int\frac{W_0(\mathbf{q})}{\hbar}\left[\frac{1}{(\omega_- - \mathbf{k}\cdot\mathbf{v}_q)} - \frac{1}{(\omega_+ - \mathbf{k}\cdot\mathbf{v}_q)}\right]\frac{d\mathbf{q}}{(2\pi)^3} = 0, \tag{15}$$

where $\omega_\pm = \omega \pm \frac{\hbar k^2}{2m}$. In what follows, we study the dispersion relation for a zero-temperature system. This allows us to describe the equilibrium in terms of δ-distributed particles:

$$W_0(\mathbf{q}) = (2\pi)^3 n_0\delta(\mathbf{q} - \mathbf{q}_0), \tag{16}$$

where n_0 is the unperturbed density and $\mathbf{q}_0$ is the net momentum of the gas, associated with the drift velocity $\mathbf{v}_0 = \hbar\mathbf{q}_0/m$. As such, Eq. (15) is then reduced to

$$1 - \left[g + Q\sqrt{n_0} - \frac{\chi}{2}k^2 + U_d(\mathbf{k})\right]\frac{n_0}{\hbar}\left[\frac{1}{(\omega_- - \mathbf{k}\cdot\mathbf{v}_0)} - \frac{1}{(\omega_+ - \mathbf{k}\cdot\mathbf{v}_0)}\right] = 0. \tag{17}$$

Rearranging terms and using the definition of the sound speed, $c_s = \sqrt{gn_0/m}$, the latter can also be written as

$$(\omega - \mathbf{k}\cdot\mathbf{v}_0)^2 = k^2c_s^2\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{\chi}{2g}k^2 + \frac{1}{g}U_d(\mathbf{k})\right] + \frac{\hbar^2k^4}{4m^2}. \tag{18}$$

The *two-stream* configuration considered in Ref. [33] can be easily accounted for here if one replaces Eq. (16) by $W_0(\mathbf{q}) = 4\pi^3 n_0[\delta(\mathbf{q} - \mathbf{q}_0) + \delta(\mathbf{q} + \mathbf{q}_0)]$. A detailed investigation of this effect, although out of the scope of this paper, would reveal interesting aspects of the phase-space properties of dipolar gases, in particular how one could explore dynamical instabilities to generate nonlinear structures, such as solitons in one-dimensional configurations [35,36].

It is instructive to compute the product of the phase velocity $v_\phi = \omega/k$ and the group velocity $v_g = \partial\omega/\partial k$. Assuming a condensate at rest ($v_0 = 0$), we obtain

$$v_\phi v_g = c_s^2\left[1 + \frac{Q}{g}\sqrt{n_0} + \frac{1}{g}U_d(\mathbf{k})\right] + 4k^2\left(\frac{\hbar^2}{4m^2} - \frac{\chi}{2g}\right). \tag{19}$$

This shows that the product $v_\phi v_g$ is nearly equal to the square of the sound speed, c_s^2, with corrections coming from the quantum (Bohm) dispersion term and from the three different processes included in the present model (dipolar potential, quantum fluctuations, and finite-energy range of close collisions). The long-range dipole-dipole interaction potential is explicitly given by [8,37,38]

$$U_d(\mathbf{R}) = \frac{C_{dd}}{4\pi}\frac{1 - 3\cos^2\theta}{|\mathbf{R}|^3}\left(\frac{3\cos^2\varphi - 1}{2}\right), \tag{20}$$

where C_{dd} is the (magnetic or electric) dipolar interaction strength, θ is the angle between the relative position $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and the direction of the external polarization field, and φ is the angle between the orientation of the dipoles and the z axis. Taking $\varphi = 0$, we obtain the Fourier transform

$$U_d(\mathbf{k}) = C_{dd}\left(\cos^2\theta_k - \tfrac{1}{3}\right), \tag{21}$$

where θ_k is the angle between the wave vector $\mathbf{k}$ and the z axis. Replacing this in Eq. (18), and assuming a condensate at rest ($\mathbf{v}_0 = 0$), we obtain the dispersion relation

$$\omega^2 = k^2c^2(\theta_k) + \frac{\hbar^2k^4}{4m^2}. \tag{22}$$

where $c(\theta_k)$ is the angle dependent Bogoliubov velocity, defined as

$$c^2(\theta_k) = c_s^2\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{k^2}{2g}\chi + \eta\left(\cos^2\theta_k - \frac{1}{3}\right)\right]. \tag{23}$$

Here, $\eta = C_{dd}/g$ is the ratio between the dipole and contact potential strength. As we can see, the dipolar interaction introduces important qualitative corrections to the characteristic sound velocity, which can become imaginary for large values of the parameter η [8]. This reflects the anisotropic nature of the dipolar potential. In particular, a critical wave number k_c

can be defined, where $\omega^2 = 0$, as

$$k_c^2 = c_s^2\left[\eta\left(\frac{1}{3} - \cos^2\theta_k\right) - \left(1 + \frac{Q}{g}\sqrt{n_0}\right)\right]\left(\frac{4m^2}{\hbar^2} - \frac{\chi}{2g}\right). \tag{24}$$

This is positive for $\theta_k \simeq \pi/2$ and $\eta \geqslant 3$ [8]. In such case, large-scale perturbations corresponding to $k \leqslant k_c$ become unstable, with a finite growth rate determined by $\omega^2 \leqslant 0$. This is physically relevant for $(k_c L) \geqslant 1$, where L is the typical size of the condensate.

Quasi-two-dimensional dipolar Bose gases

Another interesting example is the quasi-2D condensate. If a BEC is strongly confined along the z axis, with a size l_z much smaller than its transverse dimension $l_\perp$, we can still use the same WK equation, only depending on (x,y) and (k_x,k_y). In this case, g is replaced by a renormalized coupling parameter, $g_{2D} = g/(3\sqrt{2\pi}l_z)$, and the Fourier transformation of the quasi-2D dipole-dipole potential simply reads [29]

$$U_d(k) = C_{dd}F(kl_z/\sqrt{2}), \tag{25}$$

where $k = \sqrt{k_x^2 + k_y^2}$ and the function $F(x)$, with $x = kl_z/\sqrt{2}$, is defined as

$$F(x) = 1 - \tfrac{3}{2}\sqrt{\pi}x\exp(x^2)\,\mathrm{erfc}(x). \tag{26}$$

Here, we have used the complementary error function, defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt, \tag{27}$$

in terms of which the dispersion relation reads

$$\omega^2 = k^2c_{2D}^2\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{k^2}{2g}\chi + \epsilon_{dd}F(kl_z/\sqrt{2})\right] + \frac{\hbar^2k^4}{4m^2}. \tag{28}$$

Here, we have defined $c_{2D}^2 = g_{2D}(n_0/m)$ and $\epsilon_{dd} = C_{dd}/g_{2D}$. It is well known that Eq. (28) contains the roton-maxon pair [10]. In some extreme conditions, this can even lead to the formation of a supersolid, where ω^2 becomes negative for a limited interval of wave numbers k (and not over a large region $0 \leqslant k \leqslant k_c$, as in the above 3D example). In Fig. 1, we illustrate the features of Eq. (28). We observe the usual phononlike character of the spectrum for low k; for finite values of k, a roton minimum develops (see dotted line). Moreover, we can observe that the quantum LHY correction (dot-dashed line) hardens the roton mode, resulting in an overall stabilization of the system. This effect is at the origin of the dipolar droplets [11,39]. Quantum-mechanically stabilized droplets were originally introduced by Petrov in the context of Bose-Bose mixtures [40]. Conversely, the inclusion of the finite-range corrections favors the roton instability (solid line), and the spectrum exhibits an imaginary part (dashed line). If it takes place, the latter is responsible for the formation of crystalline structures [41].

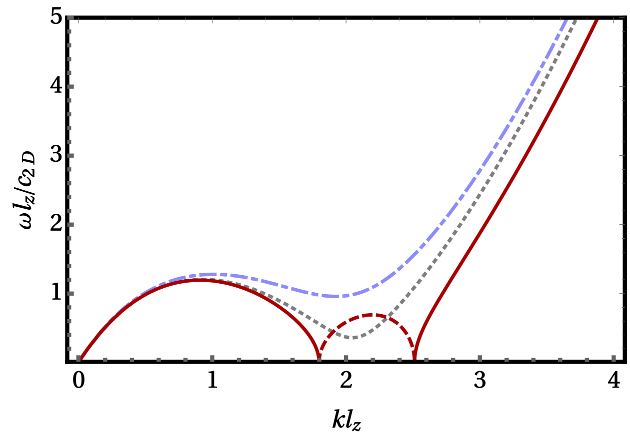

FIG. 1. Dispersion relation of elementary phonon modes in a quasi-2D dipolar condensate, displaying a roton-maxon configuration. The phase velocity $v_\phi = \omega/k$ decreases in the roton region and increases in the maxon region, thus changing the value of the corresponding Landau damping rates. The dotted line depicts the mean-field situation ($Q = 0$ and $\chi = 0$). The stabilization of the roton instability due to the LHY quantum correction is represented for $Q = 0.2g/\sqrt{n_0}$ (dot-dashed line). The inclusion of the finite range of the atom collisions enhances the roton instability, here depicted for $\chi = 0.08g/l_z^2$ (solid line). The dashed line represents the imaginary part of ω in Eq. (28). In all situations, we have set $\epsilon_{dd} = 8.2$.

IV. LANDAU DAMPING

The wave-kinetic description is particularly well suited to describe Landau damping and kinetic (phase-space) instabilities, as shown next. For that purpose, we go back to Eq. (15), which can be rewritten in the form

$$1 - \frac{g'(k)}{\hbar k}\int G_0(u)\left[\frac{1}{(u - \omega_+/k)} - \frac{1}{(u - \omega_-/q)}\right]\frac{du}{2\pi} = 0, \tag{29}$$

with $g'(k) = [g + Q\sqrt{n_0} - k^2\chi/2 + U_d(k)]$. Here, u and q represent the atom velocity and momentum components parallel to the direction of propagation, according to

$$\mathbf{v}_q = u\frac{\mathbf{k}}{k} + \mathbf{v}_\perp, \quad \mathbf{q} = q\frac{\mathbf{k}}{k} + \mathbf{q}_\perp. \tag{30}$$

We have also used the reduced distribution $G_0(q)$, such that

$$G_0(q) = \int W_0(q,\mathbf{q}_\perp)\frac{d\mathbf{q}_\perp}{(2\pi)^2}. \tag{31}$$

The integral in Eq. (29) can be decomposed as

$$\int\frac{G_0(u)}{(u - v_\pm)}du = \mathcal{P}\int\frac{G_0(u)}{(u - v_\pm)}du + i\pi G_0(v_\pm), \tag{32}$$

where $v_\pm = \omega_\pm/k$, and $\mathcal{P}$ represents the principal part of the integral, in the Cauchy sense. Inserting the latter in Eq. (29), we can interpret the dispersion relation as the zero of the *dielectric function* $\epsilon(\omega,k)$, which can be split into its real and imaginary parts, $\epsilon = \epsilon_r + i\epsilon_i$. In practice, this amounts to allowing ω to become complex, $\omega = \omega_r + i\gamma$. Explicit relations for ω_r and γ are, in general, not available, but some analytical expressions are possible in some limiting cases. In what follows, we assume that the modes are weakly damped, $|\gamma| \ll \omega_r$, a condition that we shall verify *a posteriori*. In that case, we can determine the frequency ω_r and the damping rate γ by writing [23,30]

$$\epsilon_r(\omega_r,k) = 0, \quad \gamma = -\frac{\epsilon_i(\omega_r,k)}{(\partial\epsilon_r/\partial\omega)_{\omega_r}}. \tag{33}$$

For temperatures well below the condensation temperature, we may neglect the thermal broadening in the velocity distribution. This allows us to use the cold gas condition $G_0(u) = 2\pi n_0 \delta(u)$ to obtain

$$\epsilon_r(\omega_r,k) = 1 - \frac{g'(k)k^2}{m}\frac{n_0}{\omega^2 - \hbar^2k^4/4m^2} = 0. \tag{34}$$

Restricting our discussion to soft modes only, i.e., $\omega \lesssim c_s k$, we obtain

$$\frac{\partial \epsilon_r}{\partial\omega} \simeq \frac{2g\omega}{g'(k)k^2c_s^2}. \tag{35}$$

Moreover, retaining finite temperature effects in the damping rate (33), we have

$$\gamma = \frac{gkc_s^2}{4\hbar\omega_r}\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{k^2}{2g}\chi + \frac{1}{g}U_d(k)\right]^2 \times [G_0(v_+) - G_0(v_-)]. \tag{36}$$

Such a procedure is justified as the damping rate is expected to be small (and in the case of δ-distributed systems it vanishes identically). Furthermore, in thermal equilibrium, we can assure the quantity $G_0(v_+) - G_0(v_-)$ is negative (as the distribution function decreases monotonically with respect to the particles' velocities) and the damping coefficient is negative, $\gamma < 0$. However, in out-of-equilibrium conditions, an inversion of population can eventually occur, such that $G_0(v_+) > G_0(v_-)$. In this case the excitations are kinetically unstable. It is important to notice that the quantum fluctuations (controlled by the parameter Q) and the finite-energy collisions (controlled by χ) do not change the sign of γ in Eq. (36), since the damping rate only depends on the quantity $g'(k)^2$.

A. Semiclassical limit

It is also useful to consider the semiclassical limit, valid for $|k| \ll |q|$ (long-wavelength limit). In this case, we can develop the quantities $G_0(v_\pm)$ around $v = \omega/k$, and Eq. (36) becomes

$$\gamma \simeq \frac{k^3c_s^4}{4n_0\omega_r}\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{k^2}{2g}\chi + \frac{1}{g}U_d(k)\right]^2\left(\frac{\partial G_0}{\partial v}\right)_{v=\omega/k}. \tag{37}$$

For a condensate in equilibrium at a finite temperature T, the derivative in Eq. (37) is always negative and the excitations are damped. In order to be more specific, we need an explicit expression for the reduced distribution $G_0(v)$. We shall use the Bose-Einstein distribution

$$G_0(v) = 2\pi n_0\{e^{[E(v)-\mu]\beta} - 1\}^{-1}, \tag{38}$$

where $E(v) = mv^2/2 = \hbar^2q^2/2m$, $\beta = 1/k_BT$, and the chemical potential μ provides the zero of the energy scale.

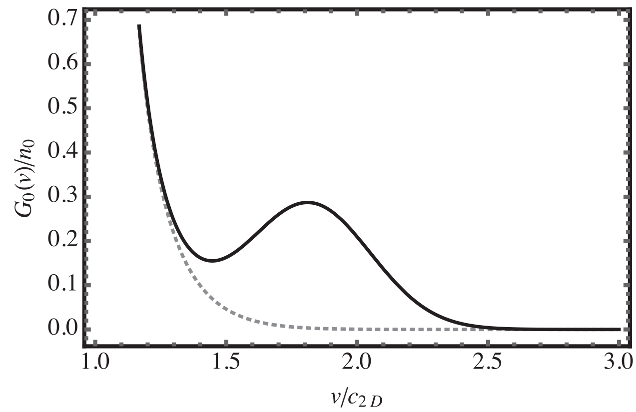

FIG. 2. Unstable reduced distribution, $G_0(v)$, of a thermal dipolar condensate in the presence of a stream of atoms, obtained from Eq. (40) for $n_b = 0.2n_0$ and $v_b = 1.7c_{2D}$ (solid line). For comparison, a stable thermal distribution is depicted (dotted line). In both cases, we have set $T = 0.4\mu/k_B$.

The Landau damping rate γ therefore reads

$$\gamma \simeq -\frac{k^3c_s^4}{4\omega_r}\left[1 + \frac{Q}{g}\sqrt{n_0} - \frac{k^2}{2g}\chi + \frac{1}{g}U_d(k)\right]^2 \times \beta e^{[E(\omega/k)-\mu]\beta}\{e^{[E(\omega/k)-\mu]\beta} - 1\}^{-2}. \tag{39}$$

We can see that γ is always negative, for all possible values of the phase velocity ω/k. However, as mentioned above, out-of-equilibrium situations may make the condensate kinematically unstable. This is linked with the possible existence of a suprathermal atomic stream, with density n_b and mean velocity v_b, as described by

$$G_0(v) = \frac{2\pi n_0}{e^{[E(v)-\mu]\beta} - 1} + \frac{2\pi n_b}{e^{\{[E(v-v_b)]-\mu\}\beta_b} - 1}, \tag{40}$$

where $\beta_b = 1/k_BT_b$, and T_b is the temperature of the suprathermal stream (see Fig. 2). In this case, the damping coefficient γ will eventually change sign, leading to an unstable region, $|v| < \omega/k \leqslant |v_b|$. This is the kinetic counterpart of the two-stream instability discussed in Ref. [33].

B. Heisenberg's uncertainty principle and the effective temperature

It is worth noticing that, even at zero temperature, the Landau damping mechanism may take place. This is a consequence of the Heisenberg uncertainty principle, which states that for a BEC with typical size $l_\perp$ the uncertainty in the atom velocity will be $\Delta v \simeq \hbar/ml_\perp$. Therefore, a finite-size $l_\perp$ is equivalent to an effective temperature $T_{\rm eff}$ of the order of Δv^2. In that case, the zero-temperature distribution $G_0(v)$ is broadened and the respective effective temperature is

$$T_{\rm eff} = \frac{1}{2k_B}\frac{\hbar^2}{ml_\perp^2}. \tag{41}$$

For a ^{164}Dy BEC, with $m \sim 162$ a.u., chemical potential $\mu \sim 3$ kHz, and $l_\perp \sim 10$–$100\ \mu$m, we obtain $T_{\rm eff} \sim 0.3$–30 pK, much less than the critical temperature $T_c \sim 30$ nK [42]. This means that Landau damping will mainly be provided by the thermal

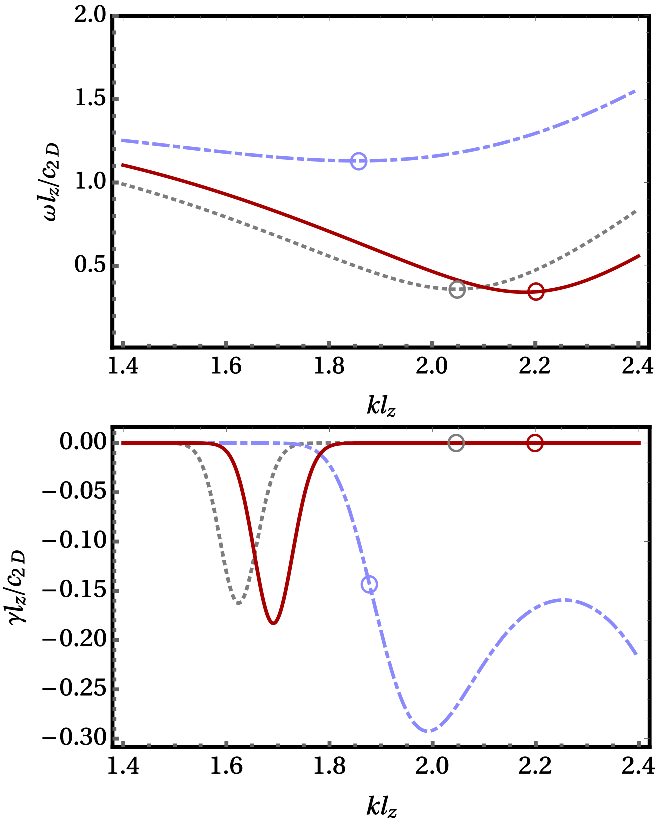

FIG. 3. Estimation of the quantum Landau damping due to the Heisenberg uncertainty. Real (top) and imaginary (bottom) parts of the dispersion relation in Eq. (28). Depicted are the mean-field case $Q = \chi = 0$ (dotted line); the LHY correction, $Q = 0.3g/\sqrt{n_0}$ and $\chi = 0$ (dot-dashed line); and its combination with the finite range of the atomic collisions, $Q = 0.3g/\sqrt{n_0}$ and $\chi = 0.08g/l_z^2$ (solid line). The circles mark the position of the roton minimum in the different situations. In all cases, we have set $\epsilon_{dd} = 8.2$ and a small effective temperature of $T_{\text{eff}} = 0.01mc_{2D}^2/k_B$, which is much less than the critical temperature for condensation T_c.

part of the condensed gas. However, in a situation where the phase velocity ω/k of the elementary excitation is strongly reduced in the vicinity of a roton minimum, Landau damping could eventually be provided by the Heisenberg broadening. This feature is illustrated in Fig. 3. As we can see, in the mean-field case (see dotted lines), Landau damping occurs below the roton minimum, while the inclusion of the quantum LHY correction displaces the Landau damping towards the roton minimum. With the inclusion of the finite range of the atomic collisions, the roton minimum remains practically undamped. This means that a balanced combination of the system size and the strength of the LHY corrections can favor Landau damping near the roton minimum, resulting in a purely quantum-mechanical suppression of the roton mode that is worthy to be investigated in future experiments.

V. TRAPPING AND DIFFUSION

As mentioned above, Landau damping results from an energy exchange between the phonon excitations and the particles in the condensate. Together with the linear analysis presented in the above sections, a more complete discussion shall include other possible kinetic effects associated with this energy exchange. The first effect we consider in this paper is known under the name of *atomic trapping*, which can take place if we consider the finite amplitude of the collective excitations in Eq. (28) [43,44]. Another is *atom diffusion*, when a large spectrum of excitations is excited in the medium [30,43]. In this case, the exchange of energy between the particles and the phonons induces diffusion in the atomic velocity space, associated with the cumulative Landau damping over the phonon spectrum, leading to second-order changes in the particle distribution. These two aspects are briefly described next. In this section, we will neglect quantum fluctuations and finite-energy effects, and take $Q = 0$ and $\chi = 0$, for simplicity. However, in the discussion that follows, the latter do not impact the physics of atom trapping and diffusion qualitatively.

Atom trapping occurs in the vicinity of wave-particle resonance, i.e., when the atom speed $v_q = \hbar q/m$ is equal to the phase velocity of the collective excitation. It can be seen that particles with momentum q satisfying the condition

$$E_{\text{res}} - g\tilde{n}_k \leqslant \frac{\hbar^2 q^2}{2m} \leqslant E_{\text{res}} + g\tilde{n}_k, \tag{42}$$

where $\tilde{n}_k$ is the phonon amplitude and $E_{\text{res}} = m\omega^2/2k^2$, can be effectively trapped by the potential created by the collective mode. In other words, the trapped particles are those with momentum q in the interval $q_- \leqslant q \leqslant q_+$, where

$$q_\pm \simeq \frac{m\omega}{\hbar k}\sqrt{1 \pm \frac{k^2 c_s^2}{\omega^2}\frac{2m^2}{\hbar^2}\left(\frac{\tilde{n}_k}{n_0}\right)}. \tag{43}$$

As we can see from Eq. (43), the potential well created by the collective excitations is particularly high near the roton minimum, for which the frequency softens. Near this region, a series of trapped states arises, with energy levels $\hbar\omega_B(1/2 + \nu)$, not exceeding $g\tilde{n}_k$, where ν is an integer. The *bounce frequency* for the trapped atoms is given by

$$\omega_B = kc_s\sqrt{\frac{\tilde{n}_k}{2n_0}\left[1 + \frac{1}{g}U_d(k)\right]}. \tag{44}$$

This trapping process is very similar to that occurring for free electrons in quantum plasmas [45,46]. In particular, we can define a similar *trapping parameter*, $\Lambda_{\text{trap}} = g\tilde{n}_k/\hbar\omega_B$, which gives the approximate number of trapped states for each mode k. For $\Lambda_{\text{trap}} \gg 1$, we are in the quasiclassical limit, and for $\Lambda_{\text{trap}} < 1/2$ trapping will be forbidden. Trapping introduces nonlinear corrections to Landau damping, which can lead to modulations of the mode amplitude at the harmonics of the bounce frequency ω_B. However, nonlinear Landau damping is outside the scope of the present paper.

Finally, we consider the case of a broad spectrum of phonons, as is the case of a turbulent BEC [47,48]. A quasi-linear theory, based on the above wave-kinetic equation, can then be established, which is formally identical to that derived in Ref. [49] for a laser-cooled gas. Each phonon excitation will be damped with the corresponding Landau damping rate, but due to global energy transfer between the particles and the turbulent field the equilibrium distribution $W_0(\mathbf{q})$ will change over a long time scale, as determined by the diffusion equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}_q\cdot\nabla-\frac{\partial}{\partial\mathbf{q}}\cdot\mathbb{D}\cdot\frac{\partial}{\partial\mathbf{q}}\right]W_0(\mathbf{q},t)=0. \tag{45}$$

Here, $\mathbb{D}$ is a diffusion tensor in the atomic velocity space, which is given by

$$\mathbb{D}=\frac{\pi}{n_0^2}\int\left[1+\frac{1}{g}U_d(\mathbf{k})\right]^2\mathbf{q}\otimes\mathbf{q}\,|\tilde{n}_k|^2\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_q)\frac{d\mathbf{k}}{(2\pi)^3}. \tag{46}$$

This expression shows that particle diffusion (leading, e.g., to the broadening of the velocity distribution) results from the sum of the trapped states mentioned above over the different Fourier components of the phonon spectrum. This particle-wave interaction is worthy to be investigated in a separate paper.

VI. CONCLUSIONS

In this paper, we have described the main properties of quantum Landau damping in dipolar condensates. The quasiclassical limit was also discussed. Our model was based on a generalized wave-kinetic equation, with a nonlocal potential, where quantum fluctuations and the finite-energy corrections were also included. We have shown that such a kinetic description is particularly adequate to describe Landau damping and kinetic instabilities associated with deviations from thermal equilibrium.

A general expression for the dispersion relation of elementary excitations in the dipolar BEC, and the corresponding Landau damping rate, were established. Typical dipolar 3D and quasi-2D configurations were also examined, which included the formation of maxon-roton pairs and the eventual occurrence of supersolids. According to our findings, Landau damping is effective near the roton minimum, and is due to a balance between the mode softening and the Heisenberg uncertainty principle, leading to a purely quantum-mechanical damping of the mode.

Additionally, we showed that the wave-kinetic formalism can be an appealing tool to discuss kinetic instabilities and nonlinear processes in the phase space. These latter effects may be particularly relevant to investigate turbulence in low-dimensional dipolar gases.

ACKNOWLEDGMENTS

J.T.M. and A.G. would like to acknowledge the financial support of CNPq Brazil. A.G. also acknowledges funding from FAPESP Brazil. H.T. acknowledges Fundação para a Ciência e Tecnologia (FCT-Portugal) Grant No. IF/00433/2015.

[1] S. Balibar, J. Low Temp. Phys. **146**, 441 (2007).
[2] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, New York, 2016).
[3] L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).
[4] D. G. Henshaw and A. D. B. Woods, Phys. Rev. **121**, 1266 (1961).
[5] M. A. Baranov, Phys. Rep. **464**, 71 (2008).
[6] C. J. Pethick and H. Smith, *Bose-Einstein Condensates in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 2008).
[7] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[8] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. **72**, 126401 (2009).
[9] T. Koch, T. Lahaye, J. Metz, B. Fröhlich, A. Griesmaier, and T. Pfau, Nat. Phys. **4**, 218 (2008).
[10] L. Santos, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. **90**, 250403 (2003).
[11] L. Chomaz, R. M. W. van Bijnen, D. Petter, G. Faraoni, S. Baier, J. H. Becher, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Nat. Phys. **14**, 442 (2017).
[12] D. H. J. O'Dell, S. Giovanazzi, and G. Kurizki, Phys. Rev. Lett. **90**, 110402 (2003).
[13] R. Mottl, F. Brennecke, K. Baumann, R. Landig, T. Donner, and T. Esslinger, Science **336**, 1570 (2012).
[14] H. Fu, Y. Wang, and B. Gao, Phys. Rev. A **67**, 053612 (2003).
[15] S. Sabari, K. Prosezian, and P. Muruganandam, Chaos, Solitons Fractals **103**, 232 (2017).
[16] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957); T. D. Lee, K. W. Huang, and C. N. Yang, *ibid.* **106**, 1135 (1957).
[17] R. Schützhold, M. Uhlmann, Y. Xu, and U. Fischer, Int. J. Mod. Phys. B **20**, 3555 (2006).
[18] A. R. P. Lima and A. Pelster, Phys. Rev. A **84**, 041604(R) (2011).
[19] U. R. Fischer, Phys. Rev. A **73**, 031602 (2006).
[20] M. Schmitt, M. Wenzel, B. Böttcher, I. Ferrier-Barbut, and T. Pfau, Nature (London) **539**, 259 (2016).
[21] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Phys. Rev. X **6**, 041039 (2016).
[22] A. Fabrocini and A. Polls, Phys. Rev. A **60**, 2319 (1999).
[23] L. Landau, J. Exp. Theor. Phys. **16**, 574 (1946); J. Phys. (USSR) **10**, 25 (1946).
[24] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, 2nd ed. (Plenum Press, New York, 1984).
[25] S. Giorgini, Phys. Rev. A **57**, 2949 (1998).
[26] M. Guilleumas and L. P. Pitaevskii, Phys. Rev. A **61**, 013602 (1999).
[27] L. P. Pitaevskii and S. Stringari, Phys. Lett. A **235**, 398 (1997).
[28] J. T. Mendonça, P. K. Shukla, and R. Bingham, Phys. Lett. A **340**, 355 (2005).
[29] S. S. Natu and R. M. Wilson, Phys. Rev. A **88**, 063638 (2013).
[30] J. T. Mendonça and H. Terças, *Physics of Ultra-Cold Matter* (Springer, New York, 2011).
[31] S. A. Gardiner, D. Jaksch, R. Dum, J. I. Cirac, and P. Zoller, Phys. Rev. A **62**, 023612 (2000).
[32] J. T. Mendonça, J. Phys. A **49**, 275501 (2016).
[33] H. Terças, J. T. Mendonça, and G. R. M. Robb, Phys. Rev. A **79**, 065601 (2009).
[34] J. Zamanian, M. Marklund, and G. Brodin, Phys. Rev. E **88**, 063105 (2013); New J. Phys. **12**, 043019 (2012).

[35] V. A. Brazhnyi, D. Novoa, and C. P. Jisha, Phys. Rev. A **88**, 013629 (2013).
[36] M. Brewczyk, M. Gajda, and K. Rzażewski, J. Phys. B: At. Mol. Opt. Phys. **40**, R1 (2007).
[37] S. Giovanazzi, A. Görlitz, and T. Pfau, Phys. Rev. Lett. **89**, 130401 (2002).
[38] R. K. Kumar, T. Sriraman, H. Fabrelli, P. Muruganandam, and A. Gammal, J. Phys. B **49**, 155301 (2016).
[39] F. Wachtler and L. Santos, Phys. Rev. A **93**, 061603(R) (2016).
[40] D. S. Petrov, Phys. Rev. Lett. **115**, 155302 (2015).
[41] N. Henkel, R. Nath, and T. Pohl, Phys. Rev. Lett. **104**, 195302 (2010).
[42] M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, Phys. Rev. Lett. **107**, 190401 (2011).
[43] D. R. Nicholson, *Introduction to Plasma Theory* (Krieger, Malabar, Florida, 1992).
[44] P. A. Damiano, J. R. Johnson, and C. C. Chaston, J. Geophys. Res. **121**, 10831 (2016).
[45] G. Brodin, J. Zamanian, and J. T. Mendonça, Phys. Scr. **90**, 068020 (2015).
[46] J. Daligault, Phys. Plasmas **21**, 040701 (2014).
[47] N. G. Parker and C. S. Adams, Phys. Rev. Lett. **95**, 145301 (2005).
[48] P. G. Kevrekidis, D. J. Frantzeskakis, and R. C.- González, *Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment*, Springer Series on Atomic, Optical, and Plasma Physics Vol. 45 (Springer-Verlag, Berlin, 2008).
[49] J. T. Mendonça, Phys. Rev. A **81**, 023421 (2010).