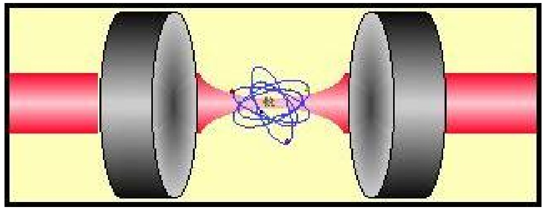
Luz e Átomos como ferramentas para Informação Quântica

Ótica Quântica

Inst. de Física

USP

Marcelo Martinelli



Lab. de Manipulação Coerente de Átomos e Luz

INCT-IQ

FAPESP

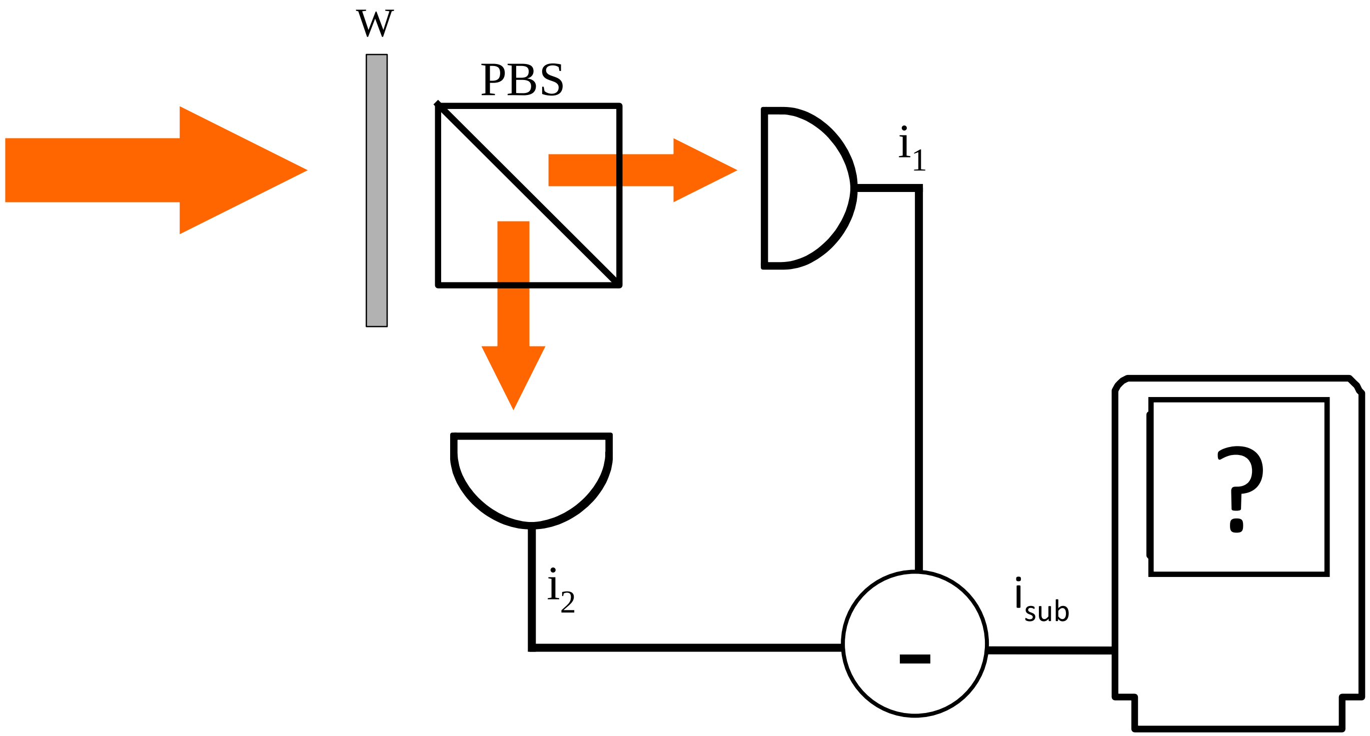
CNPq

Conselho Nacional de Desenvolvimento Científico e Tecnológico

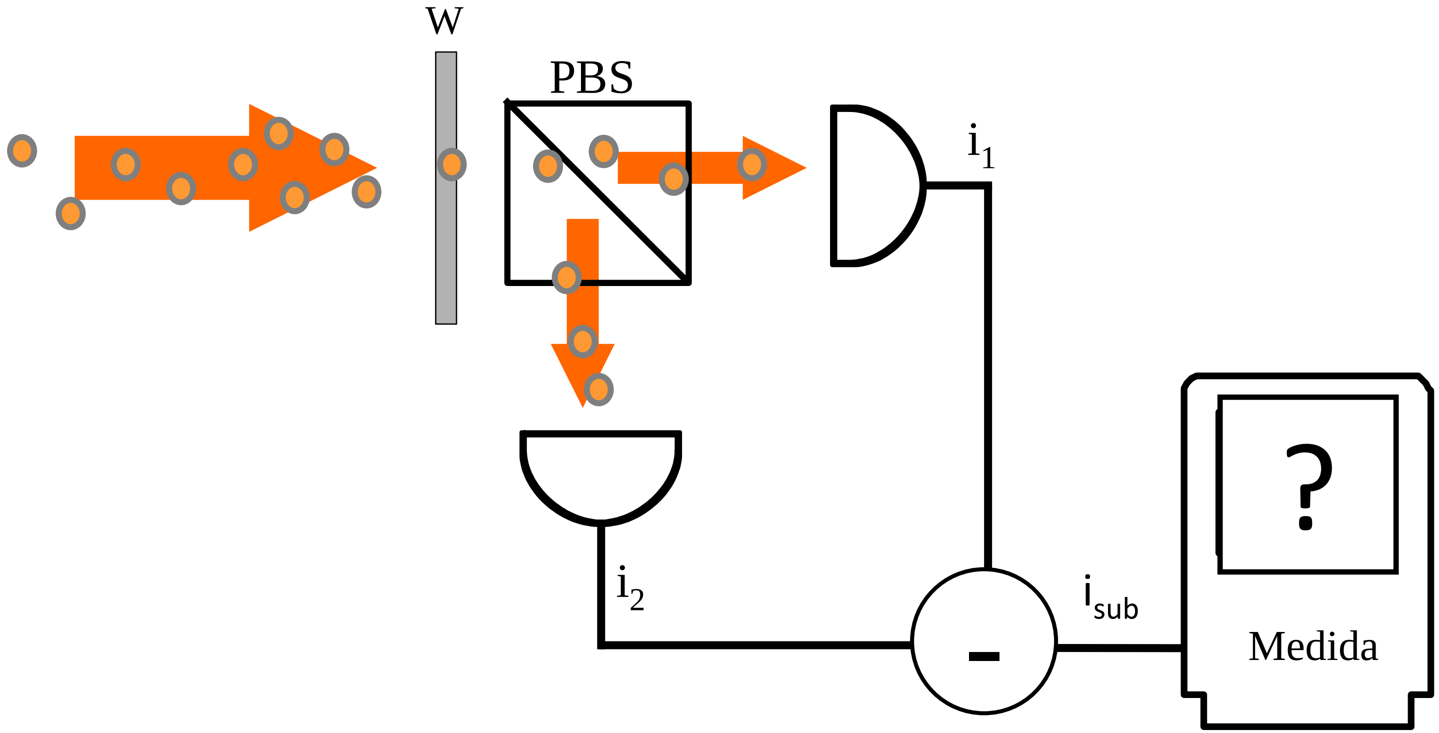
CAPES

Question:

Dividing the incident beam in two “equal” parts, what will be the result?



Answer:



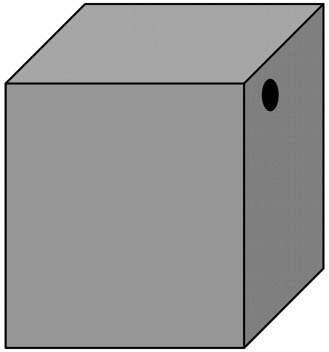
Classically: $i_{sub} = 0$

Quantically: “photons are clicks on photodetectors” (A. Zeilinger)

$$\langle i_{sub} \rangle = 0, \qquad \Delta^2 i_{sub} > 0 \,!$$

Quantum Mechanics

Birth of a revolution at the dawn of the 20th Century



Introduction of the concept of "quanta"

Energy per unit volume per unit wavelength $S_\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$

Energy per unit volume per unit frequency $S_\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3}{e^{h\upsilon/kT}-1}$

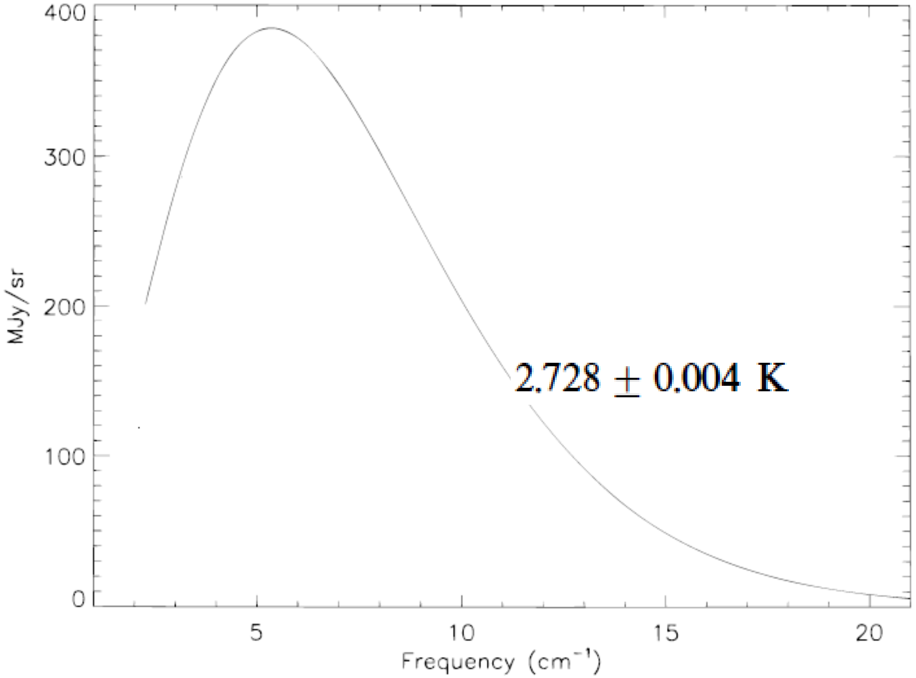


FIG. 4.—Uniform spectrum and fit to Planck blackbody (T). Uncertainties are a small fraction of the line thickness.

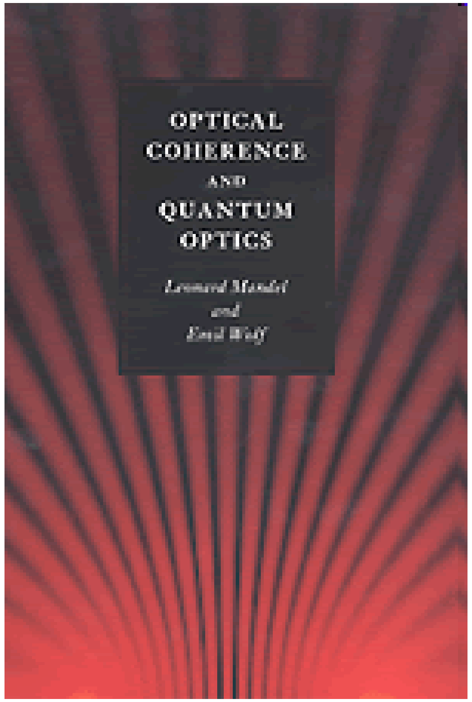
THE COSMIC MICROWAVE BACKGROUND SPECTRUM FROM THE FULL *COBE*[1] FIRAS DATA SET

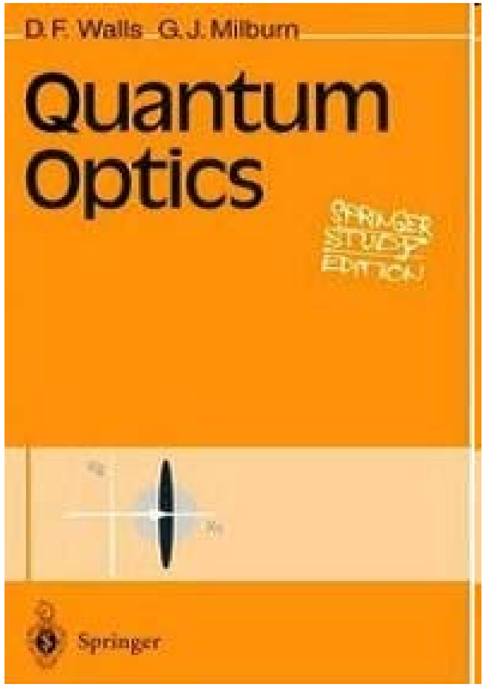
D. J. FIXSEN,[2] E. S. CHENG,[3] J. M. GALES,[2] J. C. MATHER,[3] R. A. SHAFER,[3] AND E. L. WRIGHT[4]

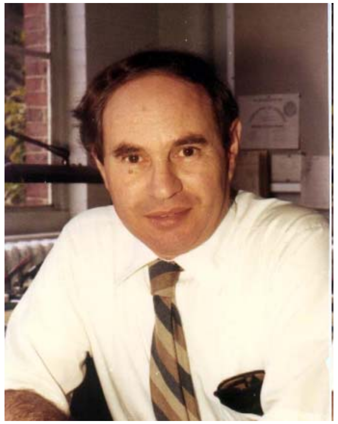
THE ASTROPHYSICAL JOURNAL, 473:576–587, 1996 December 20

Quantum Optics

Quantization of the Electromagnetic Field (on the shoulders...)







Optics

Maxwell Equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Solution in a Box

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} i\omega \left[u_{\mathbf{k}s}(t) \boldsymbol{\epsilon}_{\mathbf{k}s} e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t) \boldsymbol{\epsilon}^*_{\mathbf{k}s} e^{-i\mathbf{k}\cdot\mathbf{r}} \right],$$

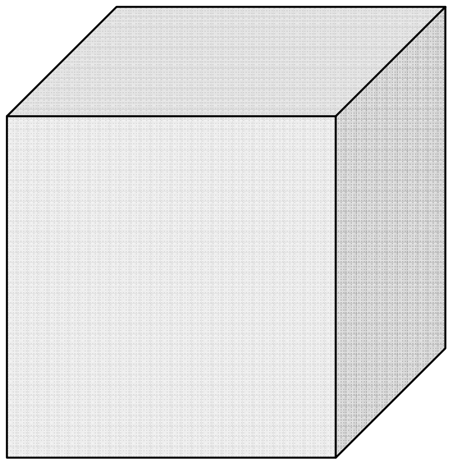
$$\mathbf{B}(\mathbf{r},t) = \frac{i}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} \left[u_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s}) e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}^*_{\mathbf{k}s}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

Wavevector

$$k_j = 2\pi n_j / L$$

Angular Frequency

$$\omega = c|\mathbf{k}|$$



Amplitude

$$u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega t}$$

Polarization

$$\boldsymbol{\epsilon}^*_{\mathbf{k}s} \cdot \boldsymbol{\epsilon}_{\mathbf{k}s'} = \delta_{ss'}$$

$$\boldsymbol{\epsilon}^*_{\mathbf{k}1} \times \boldsymbol{\epsilon}_{\mathbf{k}2} = \mathbf{k}/k$$

Optics

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2}\int_V \left[\epsilon_0 \mathbf{E}^2(\mathbf{r},t) + \frac{\mathbf{B}^2(\mathbf{r},t)}{\mu_0}\right] dv = 2\sum_{\mathbf{k}}\sum_s \omega^2 |u_{\mathbf{k}s}(t)|^2$$

Canonical Variables: going into Hamiltonian formalism

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u^*_{\mathbf{k}s}(t)$$
$$p_{\mathbf{k}s}(t) = -i\omega\left[u_{\mathbf{k}s}(t) - u^*_{\mathbf{k}s}(t)\right]$$

Optics

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{k}} \sum_{s} \left[p_{\mathbf{k}s}^2(t) + \omega^2 q_{\mathbf{k}s}^2(t) \right]$$

Canonical Variables: going into Hamiltonian formalism

$$\begin{aligned} q_{\mathbf{k}s}(t) &= u_{\mathbf{k}s}(t) + u_{\mathbf{k}s}^*(t) \\ p_{\mathbf{k}s}(t) &= -i\omega \left[u_{\mathbf{k}s}(t) - u_{\mathbf{k}s}^*(t) \right] \end{aligned}$$

Energy of the EM Field

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{s}\left[p_{\mathbf{k}s}^2(t) + \omega^2 q_{\mathbf{k}s}^2(t)\right]$$

A very familiar Hamiltonian!

Sum over independent harmonic oscillators

Energy of the EM Field

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{s}\left[\hat{p}^2_{\mathbf{k}s}(t) + \omega^2\hat{q}^2_{\mathbf{k}s}(t)\right]$$

Using creation and annihilation operators, associated with amplitudes u_{ks}

$$\hat{q}_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\omega}}\left[\hat{a}_{\mathbf{k}s}(t) + \hat{a}^\dagger_{\mathbf{k}s}(t)\right]$$

$$\hat{p}_{\mathbf{k}s}(t) = i\sqrt{\frac{\hbar\omega}{2}}\left[\hat{a}_{\mathbf{k}s}(t) - \hat{a}^\dagger_{\mathbf{k}s}(t)\right]$$

$$\left[\hat{a}_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'(t)}\right] = \delta^3_{\mathbf{k}\mathbf{k}'}\delta_{ss'}$$

$$\left[\hat{a}_{\mathbf{k}s}(t), \hat{a}_{\mathbf{k}'s'(t)}\right] = 0$$

$$\left[\hat{a}^\dagger_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'(t)}\right] = 0.$$

$$\hat{a}_{\mathbf{k}s}(t) = \hat{a}_{\mathbf{k}s}e^{-i\omega t} \qquad \hat{a}^\dagger_{\mathbf{k}s}(t) = \hat{a}^\dagger_{\mathbf{k}s}e^{i\omega t}$$

Energy of the EM Field

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_{\mathbf{k}} \left(\hat{a}^{\dagger}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} + \frac{1}{2} \right)$$

Amplitudes of Electric and Magnetic Fields

$$\hat{\mathbf{E}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \left[i\hat{a}_{\mathbf{k}s} \boldsymbol{\epsilon}_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - i\hat{a}^{\dagger}_{\mathbf{k}s} \boldsymbol{\epsilon}^{*}_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right]$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar}{2\omega\epsilon_0}} \left[i\hat{a}_{\mathbf{k}s} (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - i\hat{a}^{\dagger}_{\mathbf{k}s} (\mathbf{k} \times \boldsymbol{\epsilon}^{*}_{\mathbf{k}s}) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right].$$

Field Quadratures – Classical Description

- Classical Description of the Electromagnectic Field:

Fresnel Representation of a single mode

$$E(t)=\mathrm{Re}\{\alpha \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} i\omega \left[u_{\mathbf{k}s}(t)\boldsymbol{\epsilon}_{\mathbf{k}s} e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t)\boldsymbol{\epsilon}^*_{\mathbf{k}s} e^{-i\mathbf{k}\cdot\mathbf{r}} \right],$$

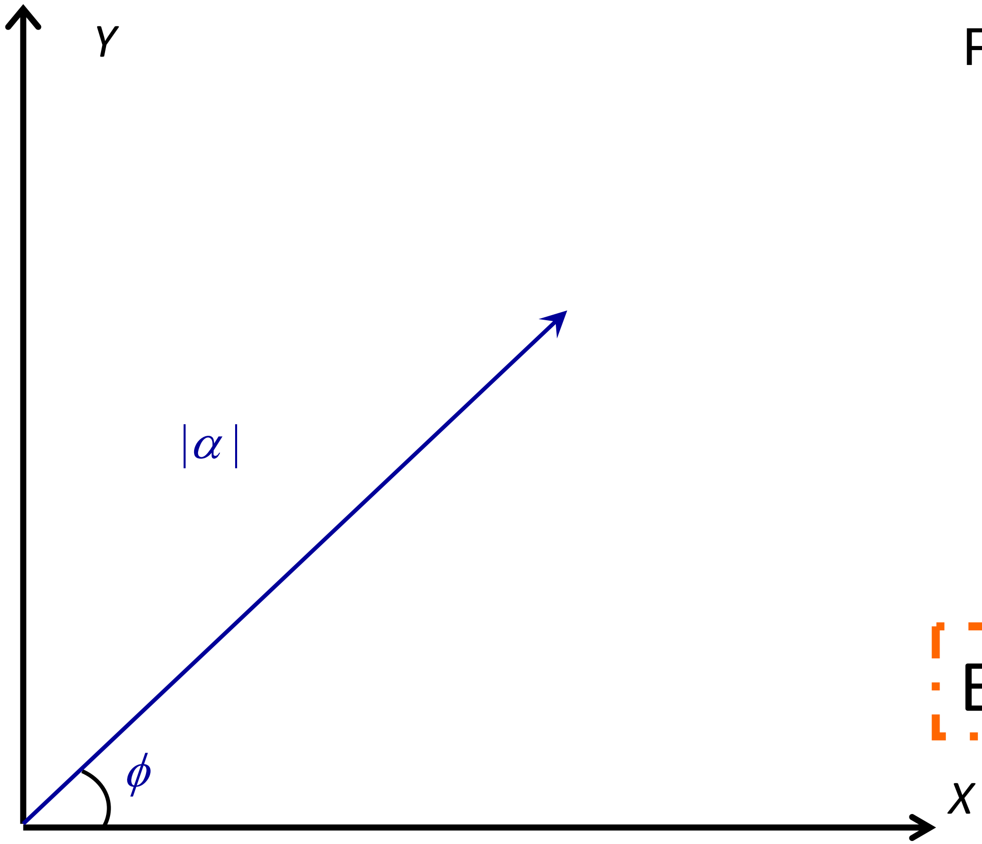
$$\mathbf{B}(\mathbf{r},t) = \frac{i}{\sqrt{\epsilon_0 L^3}} \sum_{\mathbf{k}} \sum_{s} \left[u_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s}) e^{i\mathbf{k}\cdot\mathbf{r}} - u^*_{\mathbf{k}s}(t)(\mathbf{k} \times \boldsymbol{\epsilon}^*_{\mathbf{k}s}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

Field Quadratures – Classical Description

- Classical Description of the Electromagnectic Field:

Fresnel Representation of a single mode

For a fixed position



$$E(t) = \mathrm{Re}[\alpha \exp(i\omega t)]$$

$$\alpha = X + i\,Y$$

$$E(t) = X \cos(\omega t) + Y \,\mathrm{sen}(\omega t)$$

Field Quadratures – Quantum Optics

The electric field can be decomposed as

$$\hat{\mathbf{E}}^{(+)} = \frac{i}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \left[\hat{a}_{\mathbf{k}s} \mathbf{u}_{\mathbf{k}s}(\mathbf{r}) e^{-i\omega t}\right] \qquad ; \qquad \hat{\mathbf{E}}^{(-)} = \left[\hat{\mathbf{E}}^{(+)}\right]^{\dagger}$$

And also as

$$\hat{\mathbf{E}} = \frac{2i}{L^{3/2}} \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega}{2\epsilon_0}} \epsilon \left[\hat{X} \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + \hat{Y} \sin(\mathbf{k}\cdot\mathbf{r} - \omega t)\right]$$

X and Y are the field quadrature operators, satisfying

$$\hat{X}_\theta(t) = \mathrm{e}^{-i\theta}\, \hat{a}(t) + \mathrm{e}^{i\theta}\, \hat{a}^{\dagger}(t)\ , \qquad \hat{Y}_\theta(t) = -i\left[\mathrm{e}^{-i\theta}\, \hat{a}(t) - \mathrm{e}^{i\theta}\, \hat{a}^{\dagger}(t)\right]$$

$$\left[\hat{X}(\theta), \hat{X}\left(\theta + \frac{\pi}{2}\right)\right] = 2i$$

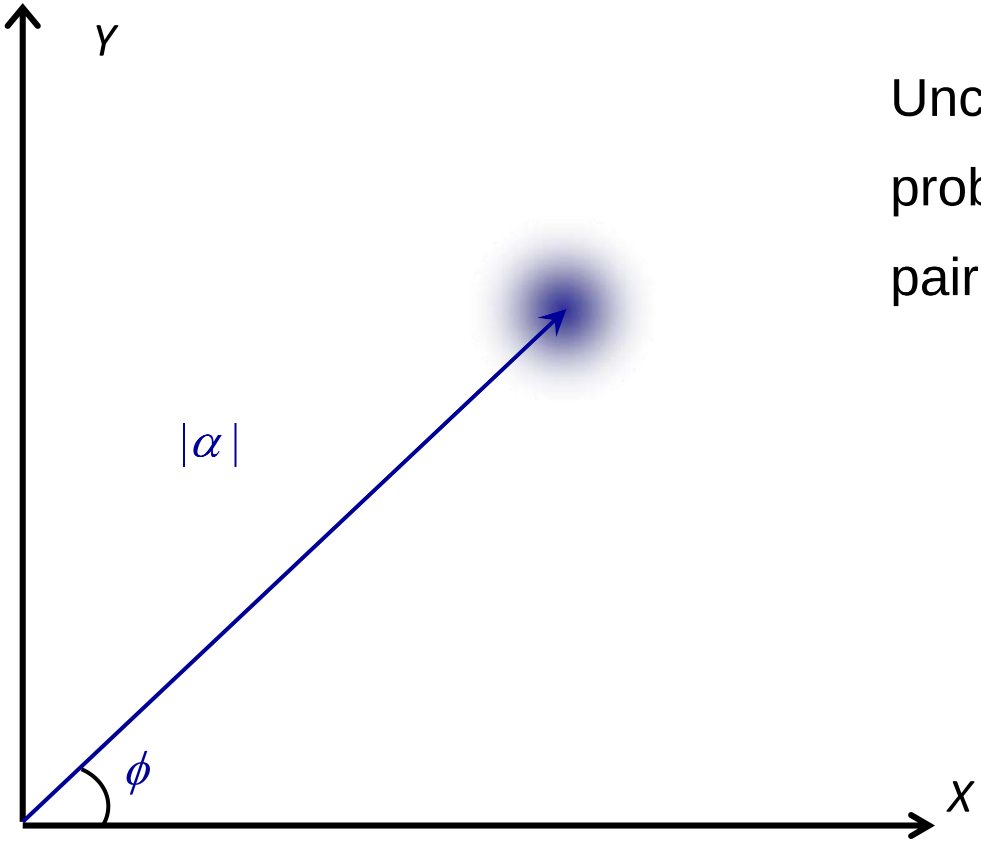
Thus,

$$\Delta X \Delta Y \geq 1$$

Field Quadratures – Quantum Optics

$$\left[\hat{X}(\theta), \hat{X}\left(\theta + \frac{\pi}{2}\right)\right] = 2i$$

Thus, $\Delta X \Delta Y \geq 1$



Uncertainty relation implies in a probability distribution for a given pair of quadrature measurements

Field quadratures behave just as position and momentum operators!

Now we know that:

- the description of the EM field follows that of a set of harmonic oscillators,
- the quadratures of the electric field are observables, and
- they must satisfy an uncertainty relation.

But how to describe different states of the EM field?

Can we find appropriate basis for the description of the field?

Or alternatively, can we describe it using density operators?

And how to characterize these states?

Quantum Optics – Number States

Eigenstates of the number operator

$$\hat{n}_{\mathbf{k}s} = \hat{a}^{\dagger}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} \qquad \hat{n}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$$

Number of excitations in a given harmonic oscillator →

number of excitations in a given mode of the field →

number of photons in a given mode!

Annihilation and creation operators:

$$\hat{a}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}}|n_{\mathbf{k}s}-1\rangle,$$

$$\hat{a}^{\dagger}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}+1}|n_{\mathbf{k}s}+1\rangle,$$

$$\hat{a}_{\mathbf{k}s}|0\rangle = 0.$$

Fock States:

Eigenvectors of the Hamiltonian

$$|\{n\}\rangle = \prod_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$$

$$\hat{\mathcal{H}}|\{n\}\rangle = \left[\sum_{\mathbf{k}s}(n_{\mathbf{k}s}+1/2)\hbar\omega\right]|\{n\}\rangle$$

$$\mathcal{E} = \sum_{\mathbf{k}s}\left[\hbar\omega_{\mathbf{k}}\left(\hat{n}_{\mathbf{k}}+\frac{1}{2}\right)\right]$$

Quantum Optics – Number States

Complete, orthonormal, discrete basis

$$\langle n_{\mathbf{k}s}|m_{\mathbf{k}s}\rangle = \delta_{n_{\mathbf{k}s}m_{\mathbf{k}s}} \Rightarrow \langle\{n\}|\{m\}\rangle = \prod_{\mathbf{k}s}\delta_{n_{\mathbf{k}s}m_{\mathbf{k}s}},$$

$$\sum_{n_{\mathbf{k}s}=0}^{\infty}|n_{\mathbf{k}s}\rangle\langle n_{\mathbf{k}s}| = 1 \Rightarrow \sum_{\{n\}}|\{n\}\rangle\langle\{n\}| = 1.$$

Disadvantage: except for the vacuum mode it is quite an unusual state of the field.

Can we find something better?

Quantum Optics – Coherent States

Eigenvalues of the annihilation operator: $a_{\mathbf{k}s}|\alpha_{\mathbf{k}s}\rangle = \alpha_{\mathbf{k}s}|\alpha_{\mathbf{k}s}\rangle$

In the Fock State Basis: $$|\alpha_{\mathbf{k}s}\rangle = e^{-|\alpha_{\mathbf{k}s}|^2/2} \sum_{n_{\mathbf{k}s}=0}^{\infty} \frac{\alpha_{\mathbf{k}s}^{n_{\mathbf{k}s}}}{\sqrt{n_{\mathbf{k}s}!}}|n_{\mathbf{k}s}\rangle$$

Completeness:

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = 1$$

but is not orthonormal

$$\langle\alpha|\alpha'\rangle = exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right)$$

Over-complete!

Moreover:

- corresponds to the state generated by a classical current,
- reasonably describes a monomode laser well above threshold,
- it is the closest description of a “classical” state.

Quantum Optics – Number States

Precise number of photons

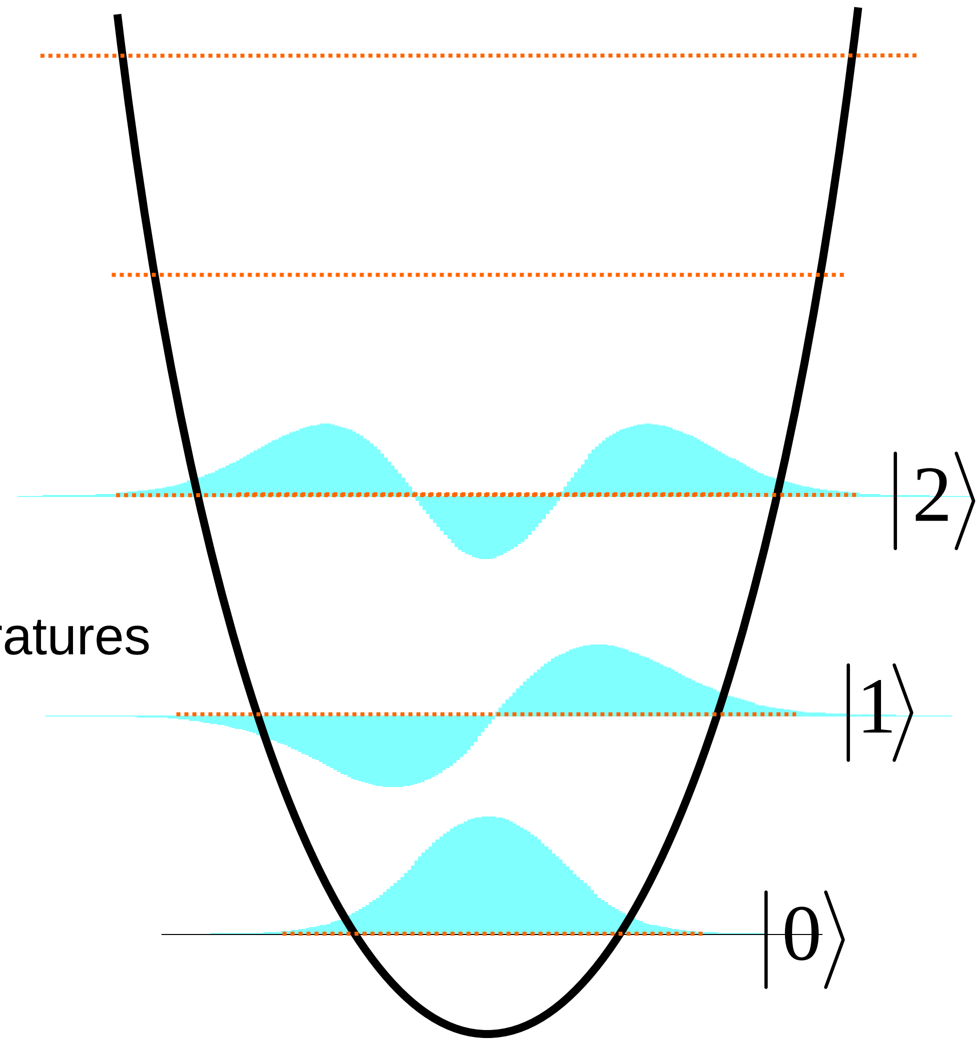
$$\langle \hat{n} \rangle = n$$

$$\Delta \hat{n}^2 = 0$$

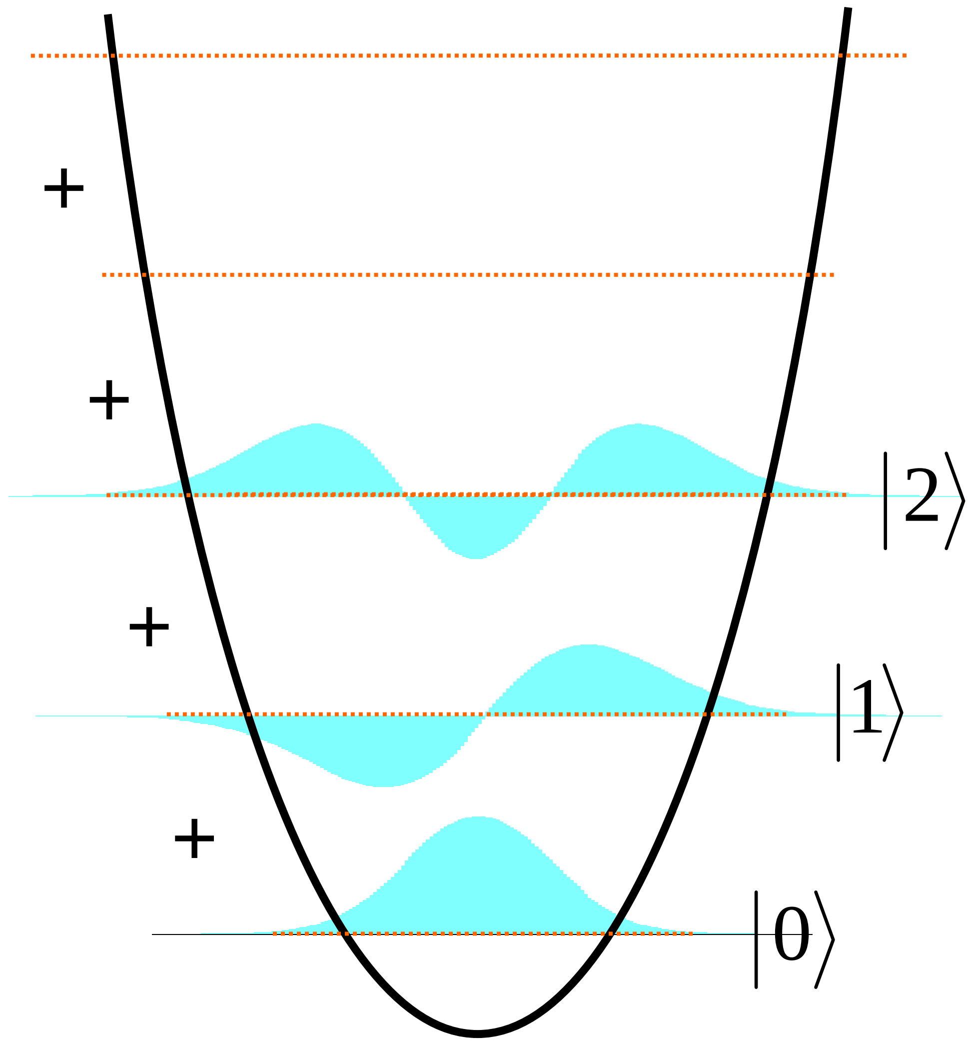
Growing dispersion of the quadratures

$$\langle \hat{X} \rangle = \langle \hat{Y} \rangle = 0$$

$$\langle \hat{X}^2 \rangle = \langle \hat{Y}^2 \rangle = 2n + 1$$

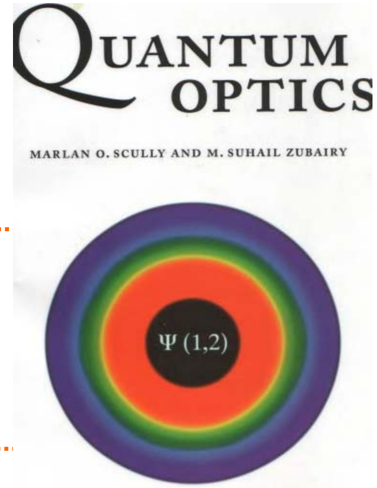


Quantum Optics – Coherent State



$$|\alpha_{\mathbf{k}s}\rangle = e^{-|\alpha_{\mathbf{k}s}|^2/2} \sum_{n_{\mathbf{k}s}=0}^{\infty} \frac{\alpha_{\mathbf{k}s}^{n_{\mathbf{k}s}}}{\sqrt{n_{\mathbf{k}s}!}} |n_{\mathbf{k}s}\rangle$$

Quantum Optics – Coherent State



$$|\alpha\rangle = D(\alpha)|0\rangle$$

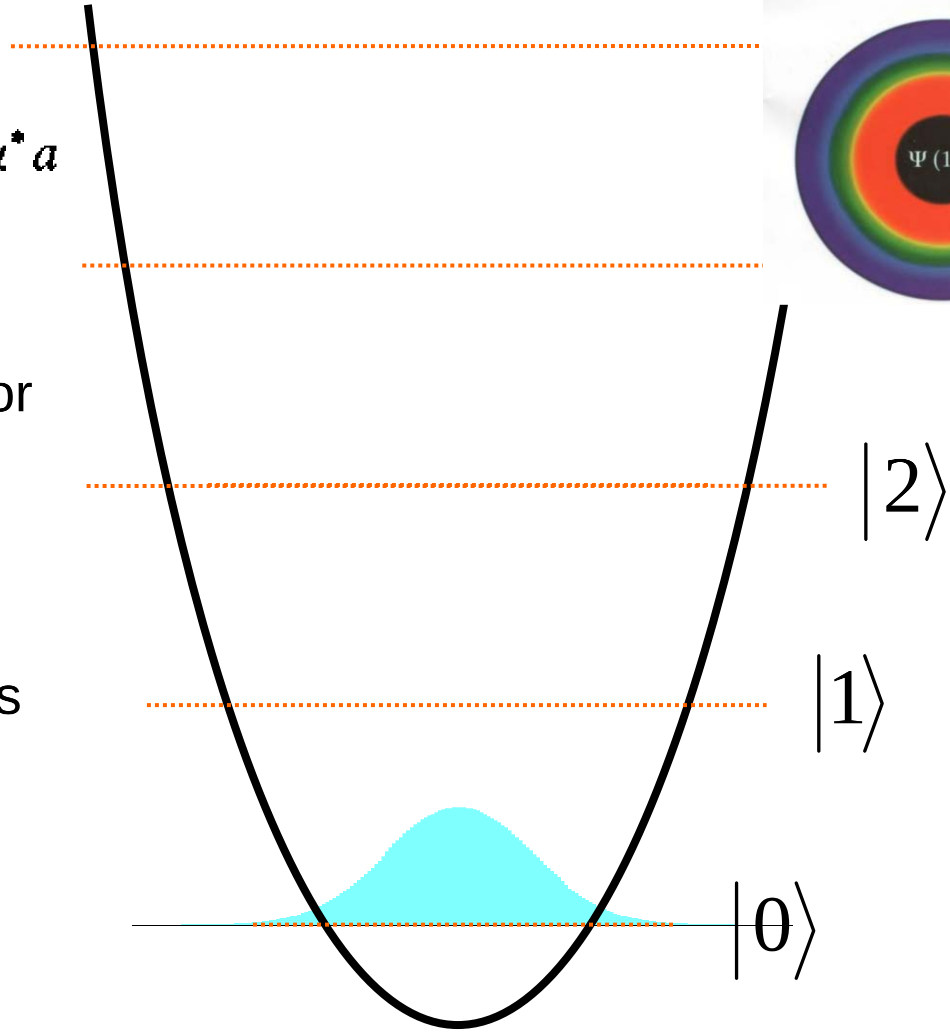
$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

Mean value of number operator

$$\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$$

Poissonian distribution of photons

$$p(n) = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}$$



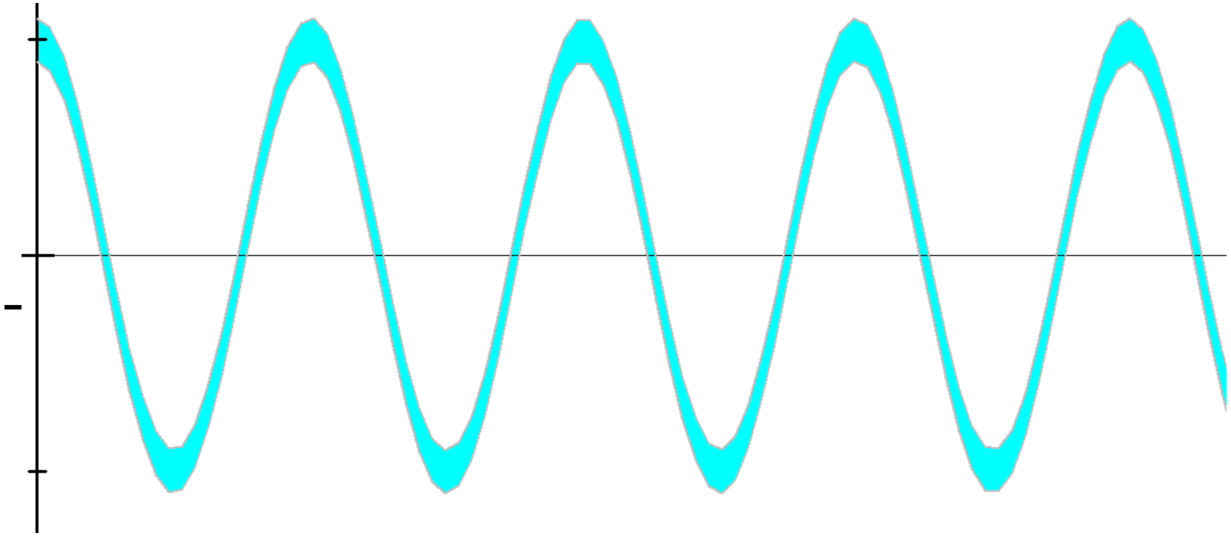
Therefore, variance of photon number is equal to the mean number!

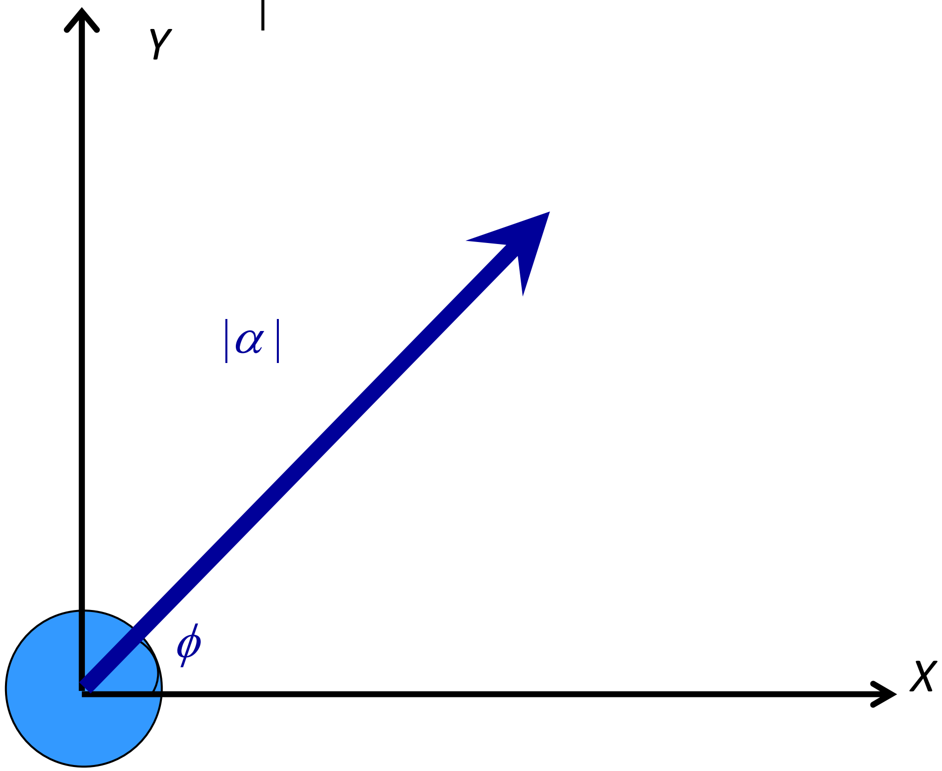
$$\Delta^2\hat{n} = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 = |\alpha|^2$$

Quantum Optics – Coherent State

$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\alpha) = \exp(\alpha a^{\dagger} -$$





Quantum Optics – Coherent Squeezed States

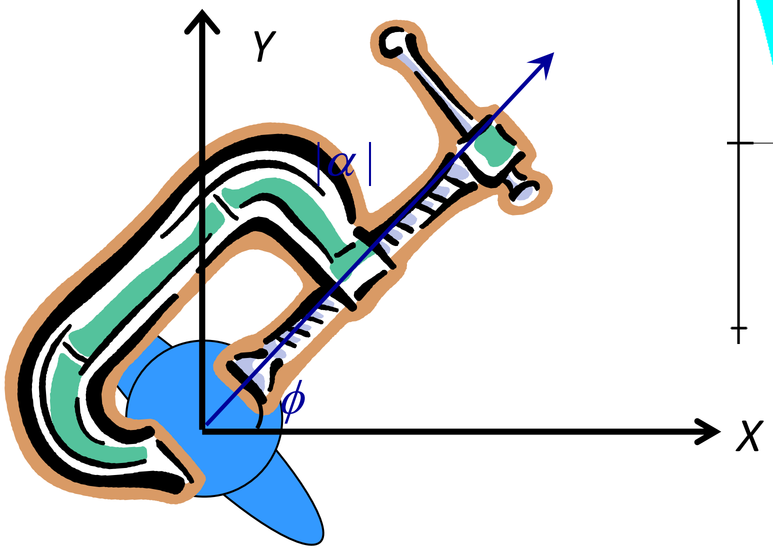
$$|\alpha\rangle = D(\alpha)|0\rangle$$

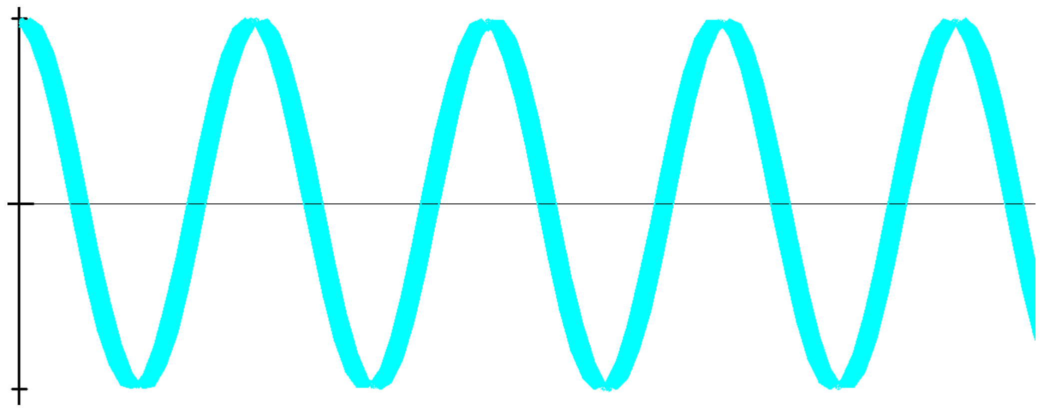
$$D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*} a)$$

$$S(\varepsilon) = \exp(1/2\varepsilon^{*} a^{2} - 1/2\varepsilon a^{\dagger\,2})$$

$$\varepsilon = r e^{2i\phi}$$

$$|\alpha, \varepsilon\rangle = D(\alpha) S(\varepsilon)|0\rangle$$





Quantum Optics – Density Operators

Statistical mixture of pure states

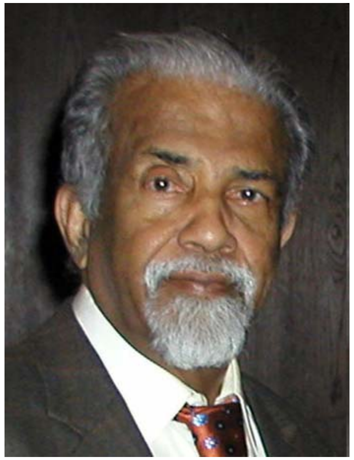
$$\rho = \sum p_k \rho_k \qquad \sum p_k = 1 \qquad \rho_k = |\psi_k\rangle\langle\psi_k|$$

Coherent States $|\alpha\rangle$

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha$$

P(α) : representation of the density operator:

Glauber and Sudarshan



Quantum Optics – Density Operators

Coherent States $|\alpha\rangle$ $$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha$$

P(α) : representation of the density operator:

Glauber and Sudarshan

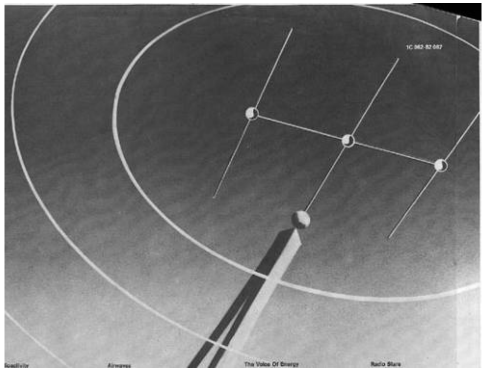
Representations of the density operators provide a simple way to describe the state of the field as a function of dimension 2N, where N is the number of modes involved.

P representation is a good way to present “classical” states, like thermal light or coherent states.

But it is singular for “non classical states” (e.g. Fock and squeezed states).

We will see some other useful representations, but for the moment, how can we get information from the state of the field?

Quantum Optics – Measurement of the Field



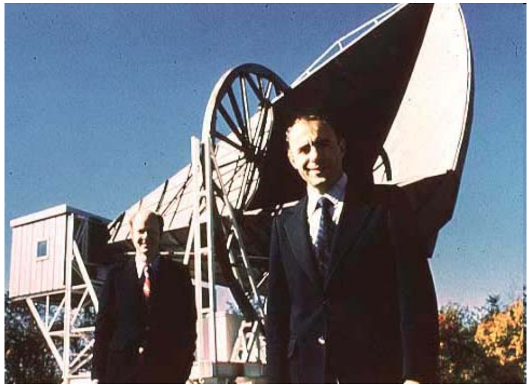
Slow varying EM Field can be detected by an antenna:

→ conversion of electric field in electronic displacement.

→amplification, recording, analysis of the signal.

→electronic readly available.

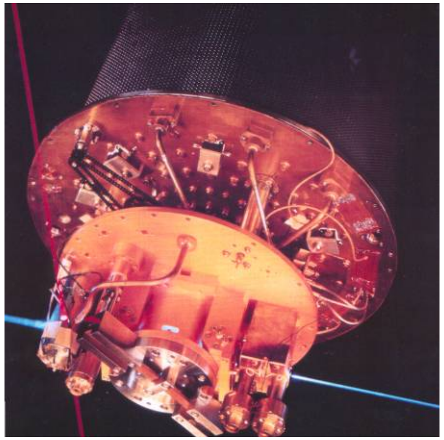
Example: 3 K cosmic background (Penzias & Wilson).

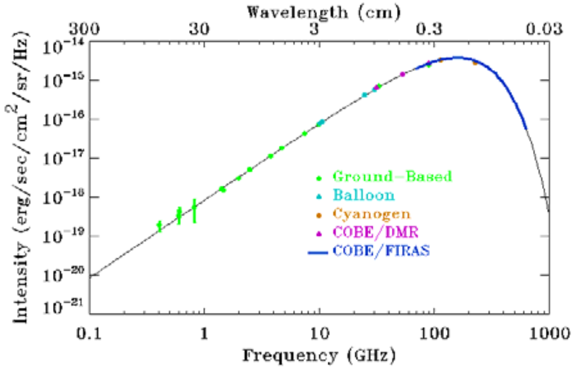


Problems:

→Even this tiny field accounts for a strong photon density.

→Every measurement needs to account for thermal background (e.g. Haroche *et al.*).





Quantum Optics – Measurement of the Field

Fast varying EM Field cannot be measured directly.

We often detect the mean value of the Poynting vector: $\mathbf{S} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$

Photoelectric effect converts photons into ejected electrons

We measure photo-electrons

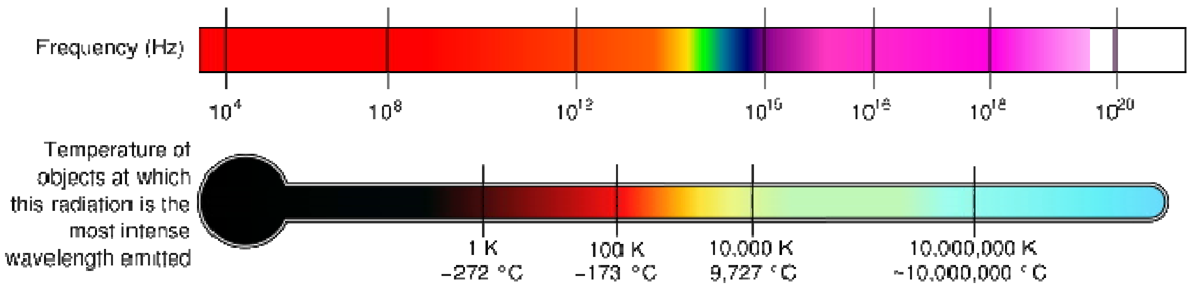
→individually with APDs or photomultipliers – a single electron is converted in a strong pulse – discrete variable domain,

→in a strong flux with photodiodes, where the photocurrent is converted into a voltage – continuous variable domain.

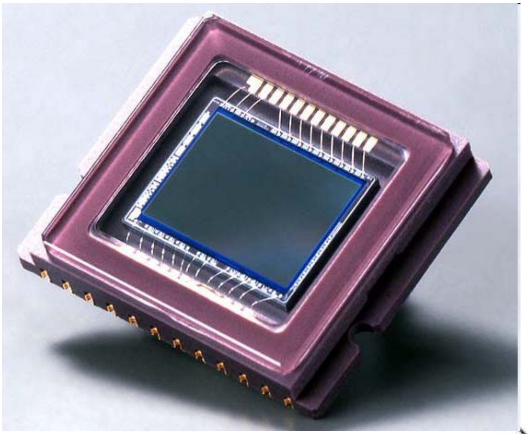
Advantages: in this domain, photons are energetic enough:

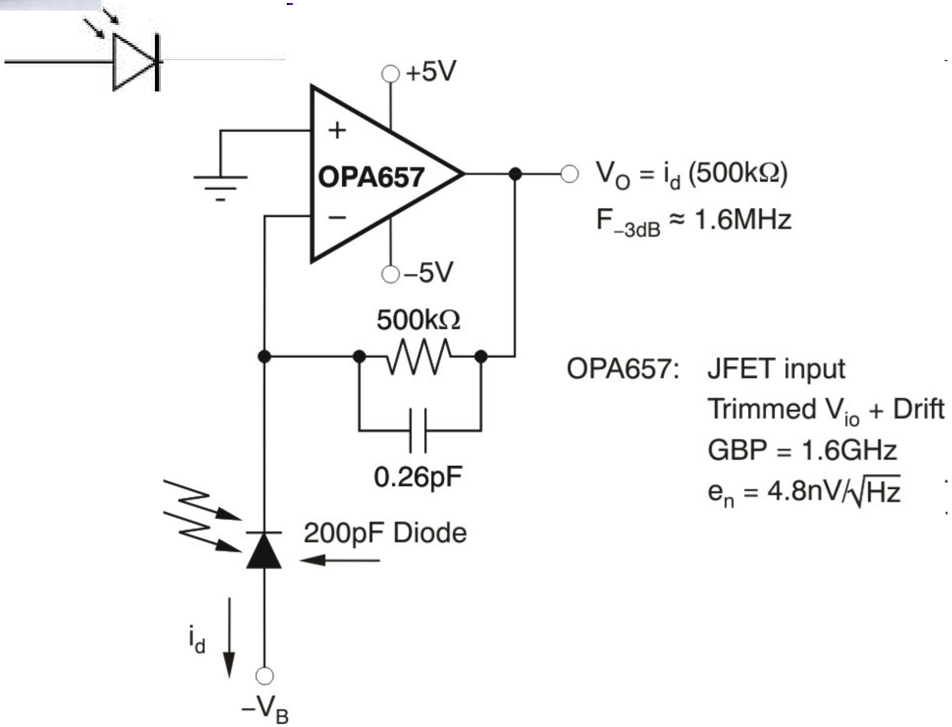
→in a small flux, every photon counts.

→for the eV region (visible and NIR), presence of background photons is negligible: measurements are nearly the same in L-He or at room temperature.



And detectors are cheap!

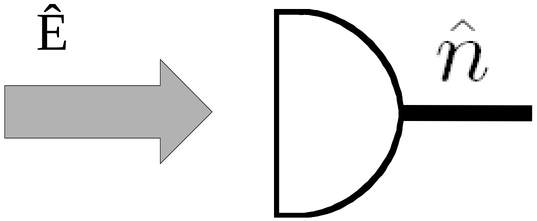




Quantum Optics – Measurement of the Intense Field

We can easily measure photon flux: field intensity

(or more appropriate, optical power)



$$I = \langle E^* \, E \rangle = \alpha^* \, \alpha$$

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

$$\hat{a}^\dagger = \alpha + \delta \hat{a}^\dagger$$

$$\alpha = |\alpha| exp(i\varphi)$$

$$\hat{n} = |\alpha|^2 + |\alpha| e^{i\varphi} \delta \hat{a}^\dagger + |\alpha| e^{-i\varphi} \delta \hat{a} + \delta \hat{a}^\dagger \delta \hat{a}$$

$$\hat{n} = |\alpha|^2 + |\alpha| \delta \hat{p} + O(2)$$

Quantum Optics – Measurement of the Intense Field

OK, we got the amplitude measurement, but that is only part of the history!

Amplitude is directly related to the measurement of the number of photon, (or the photon counting rate, if you wish).

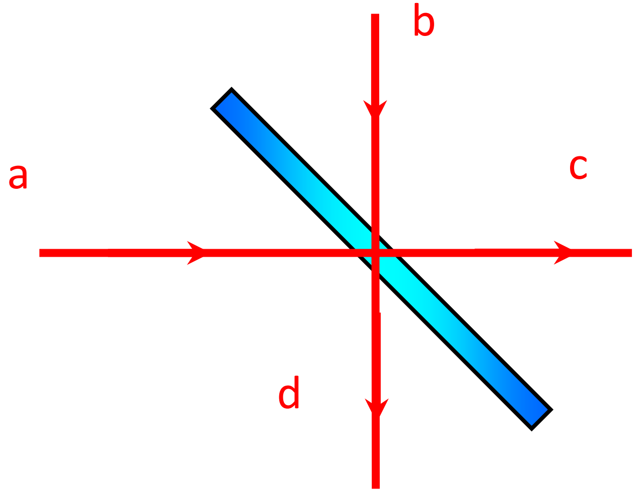
This leaves an unmeasured quadrature, that can be related to the phase of the field.

But there is not such an evident “phase operator”!

Still, there is a way to convert phase into amplitude: interference and interferometers.

Michelson or Mach Zender demonstration

Building an Interferometer – The Beam Splitter



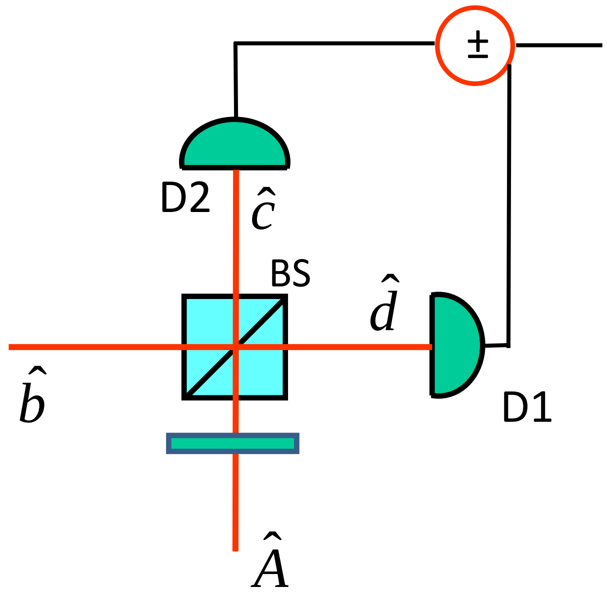
$$\hat{c} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{b}\right)$$

$$\hat{d} = \frac{1}{\sqrt{2}}\left(\hat{b} - \hat{a}\right)$$

$$\hat{n}_c = \frac{\hat{n}_a + \hat{n}_b + \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}}{2}$$

$$\hat{n}_d = \frac{\hat{n}_a + \hat{n}_b - \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}}{2}$$

Building an Interferometer – The Beam Splitter



$$\hat{n}_c = \frac{\hat{n}_a + \hat{n}_b + \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}}{2}$$

$$\hat{n}_d = \frac{\hat{n}_a + \hat{n}_b - \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}}{2}$$

$$\hat{n}_+ = \hat{n}_a + \hat{n}_b$$

$$\hat{n}_- = \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}$$

Homodyning

if $< |\hat{a}| > \; << \; < |\hat{b}| >$

$$\hat{n}_-(t) = |\beta| \left(\hat{A}(t) e^{-i\theta} + \hat{A}^\dagger(t) e^{i\theta} \right)$$

Vacuum Homodyning

$$\hat{n}_+ = \hat{n}_b$$

$$\langle \hat{n}_- \rangle = 0$$

$$\Delta^2 \hat{n}_- = \langle \hat{n}_b \rangle$$

Calibration of the Standard Quantum Level

$$\begin{aligned}
\Delta^2 \hat{n} &= \langle (\hat{n} - \langle \hat{n} \rangle)^2 \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\
&= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 = \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 + \langle \hat{a}^\dagger \hat{a} \rangle \\
&= \underbrace{\langle : \hat{n}^2 : \rangle - \langle \hat{n} \rangle^2}_{\text{"Classical" Variance}} + \underbrace{\langle \hat{n} \rangle}_{\text{Shot noise !}}
\end{aligned}$$

“Classical” Variance

Shot noise !

Vacuum Homodyning allows the calibration of the detection, producing a Poissonian distribution in the output (just like a coherent state).

$$|\alpha\rangle = exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$p_n = |\langle \hat{n} | \alpha \rangle|^2 = exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{n!}$$