Appendix to the “Variable Friction Force” Experiment Guide

I. Signs in the Equation of Motion

In the figures below, the *x*O*y* reference frame was chosen according to the guide of **Part I**, so that the projection O*x* of the coin's velocity is always positive, while in the *y* direction it starts positive, decreases and becomes negative, with an increasing absolute value after the point of maximum height ($v_y = 0$). **Figures 1** and **2** below show the force diagram viewed from the side and from above the inclined plane, respectively. Since the coin moves only on the surface of the inclined plane, the forces in the direction perpendicular to the plane are balanced, i.e., $N = mg\cos\theta$ all along the movement. The resistive force depends on the speed, so the diagram when the coin goes up the plane is different from that of the descent.

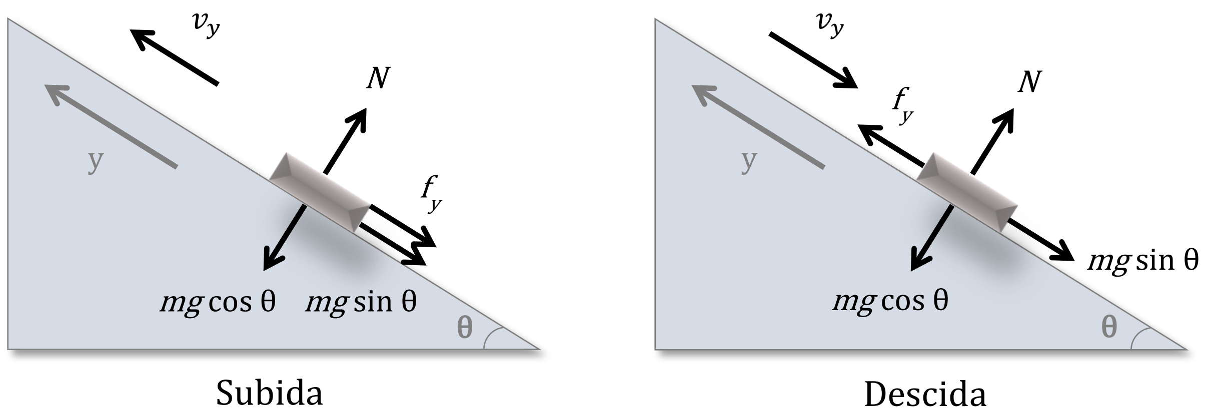

Figure 1. Free body diagram of the coin looking the plane sideways. The symbols *mg*, *N* and f_y represent the weight, the normal force and the *y* component of the resistive force, respectively, and θ, the angle between the horizontal and inclined planes.

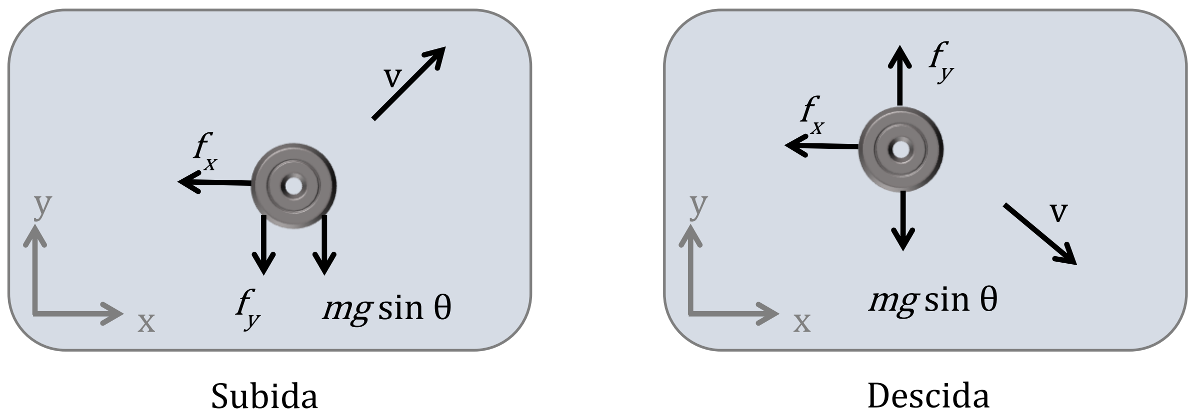

Figure 2. Top view of the free body diagram of the coin. The symbol f_x represents the component *x* of the resisting force; the other symbols are defined in the legend of **Fig. 1**.

The equation of motion is obtained from Newton's 2nd law

$$\vec{F}_R = m \cdot \vec{a},$$

where $\vec{F}_R = F_x\hat{\imath} + F_y\hat{\jmath}$ is the resultant force and $\vec{a}$ is the acceleration. Its projections in the directions *x* and *y* are:

$$\begin{cases} F_x = m \cdot a_x \\ F_y = m \cdot a_y \end{cases}$$

In this experiment, the acceleration is measured and the resultant force is deduced, in order to isolate the resistive force from it, since the weight is constant. Thus, the signs of the resistive force components, expressed as:

$$\vec{f} = f_x\hat{\imath} + f_y\hat{\jmath}$$

will come from the algebraic properties of the equations. From the free body diagram of figs. 1 and 2, the resultant force has components:

$$F_x = f_x$$

$$F_y = f_y - mg\sin\theta$$

where the magnitude of the local gravitational acceleration is represented by the constant $g > 0$. The negative sign of the weight component is necessary to correctly express its orientation in the chosen frame of reference – it is necessary to assign the proper sign to each data that is a vector quantity so that, after algebraic manipulations, the deduced quantities have the correct signs.

Substituting the above expressions for the resultant force into Newton's 2^{nd} law projections and isolating the components of the resisting force, one obtains:

$$f_x = m \cdot a_x$$

$$f_y = F_y + mg\sin\theta = m \cdot a_y + mg\sin\theta$$

as appears in the guide of Part I, eqs. (7) and (8).

Hence, f_x will have the sign of a_x – always negative, since, from the measurement of the position in the images, it is verified that the horizontal movement is always delayed. Now, the sign of f_y depends on the relationship between a_y and $g\sin\theta$, expected to be always opposite to the sign of the *velocity* v_y, that **does not enter** in this expression.

These equations transform the accelerations deduced from the positions into forces, which are needed to model the mechanics of this motion, therefore the embedded signals are critical. The reference system must be considered in order to correctly adopt these signals and interpret the results.

II. Resistant force properties from the data

Items **B15** and **B16** of the guide of Part II propose to verify if the experimental data obtained from the recorded movement support Amontons' laws on contact friction, which define the magnitude and direction of the friction force. If the experimental data do not contradict these laws, they will serve as the basis for a theoretical model of the system that will make it possible to predict the trajectory of the coin along the surface of the plane. In the next items, we will use the notation adopted in the guide for Part II.

Test of Hypothesis I: **the resistive force magnitude is constant.**

Amontons' laws state that the magnitude of the kinetic friction force, f_{at}, is constant and proportional to the normal force, N,

$$f_{at} = \mu_c N \qquad (1)$$

where the proportionality constant μ_c is the coefficient of kinetic friction. From the balance of forces in the direction normal to the plane, it can be deduced that the normal component of the contact force between the coin and the plane is constant in this experimental arrangement, with modulus:

$$N = mg\cos\theta \tag{2}$$

From the components of the resistive force at each instant, its intensity can be obtained with:

$$f = \sqrt{f_x^2 + f_y^2} \tag{3}$$

Therefore, if the resistive force can be interpreted as the kinetic frictional force ($\vec{f} = \vec{f}_k$), we can substitute eqs. (2) and (3) in eq. (1) and expect the following equality to be valid in the experiment:

$$\sqrt{f_x^2 + f_y^2} = \mu_c\, mg\cos\theta \tag{4}$$

Figure 3 presents the experimental values of the resistive force modulus as a function of time for one of the data sets. The standard deviation of the distribution is $\sigma = 279$ g·cm/s² and the mean value of the dataset and its standard deviation are $\bar{f} = 826(54)$ g·cm/s².

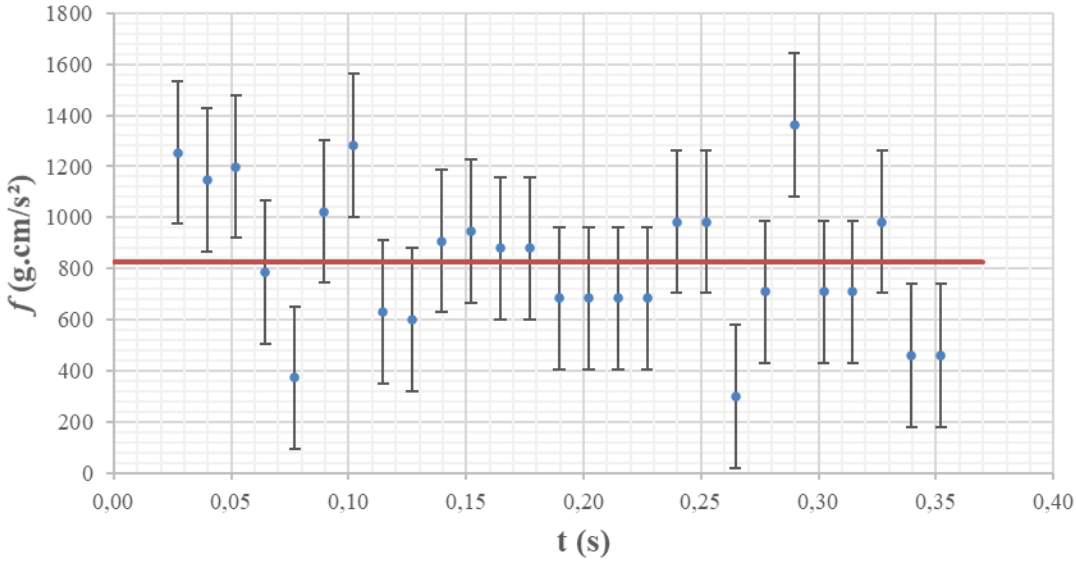

Figure 3. Experimental values of the resistive force modulus as a function of time, obtained from equation (3). The uncertainty bars correspond to a standard deviation and the solid line was drawn on the mean value of the experimental data.

When a quantity has a constant value and the data obtained have a distribution that obeys a Gaussian probability function, about 68% of the measured values will be within a standard deviation of the mean value, about 95% of them will be within two deviations; probably none to more than three deviations, unless you take a lot of data (on the order of a hundred or more). This is exactly what happens by looking at the graph in **Fig. 3** – 18 of the 27 points are less than one standard deviation from the mean value and none of them are more than two standard deviations. Therefore, it is plausible to adopt the modulus of the resistive force as a constant and to adopt the mean value of this distribution to the first estimate of the coefficient of friction from eq. (4), $\mu_e = 0{,}29(2)$.

Test of Hypothesis II: *Resistive force is opposite to velocity***.**

Amontons' laws establish that the friction force direction is opposite to the velocity, so the angle between these vectors is equal to π rad at any time. One way to validate this property for the resistive force is to verify the compatibility of the experimental values of this angle with this expected value. Unfortunately, there is no simple way to perform this analysis. Below we explain one of the possible methods, which simplifies the statistical interpretation of the result.

The angle between two vectors can be deducted from the cross product, which in this plane motion is:

$$\vec{v} \times \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & 0 \\ f_x & f_y & 0 \end{vmatrix} = (v_x f_y - v_y f_x)\hat{k} \qquad (5)$$

where $\hat{k} = \hat{\imath} \times \hat{\jmath}$. The expression of the modulus of the cross product allows us to link this result to the angle:

$$|\vec{v} \times \vec{f}| = v \cdot f \cdot \sin \alpha', \qquad \text{with } \alpha' \in [0; \pi] \qquad (6)$$

In eq. (5), the component $\hat{k}$ can be positive, null or negative, while eq. (6) defines α', the *smallest* angle between the vectors, always in the range $[0; \pi]$, because sen $\alpha' \geq 0$, since all other quantities in this equation are positive or zero.

According to Amonton's laws, the velocity and resisting force vectors are opposite, so the angle α will be compatible with π rad, if it's true. However, the measured angle will be close to, but likely different from, this value due to statistical fluctuation in the experimental data. Since expression (6) only allows values equal to or less than π, it will be difficult to estimate the true value of the angle from the histogram of the angles measured at different times. The way to solve this is to set the angle measurement reference to the velocity and determine the angle α that the velocity vector has to rotate, always around the same axis and counterclockwise, to point in the direction of the resistive force vector (see **Fig. 4**). Measured in this way, the experimental data should be distributed around the value π, and the average of these data will constitute an estimate of the true value of the angle between the vectors. **Figure 4** shows the possible relative orientations of the velocity and the measured resistive force, which define the vertical component of the cross product $\vec{v} \times \vec{f}$, that is, they define the sign of the value calculated by eq. (5). It is necessary to distinguish the quadrants of the reference system that have the abscissa in the direction of the velocity, represented with dashed lines and coupled to the velocity, and those of the reference system *x*O*y*.

The angle α that the velocity vector $\vec{v}$ needs to rotate counterclockwise around the axis in the direction $\vec{k}$ to point in the direction of $\vec{f}$, as illustrated by **Fig. 4**, follow the relationship:

$$v_x f_y - v_y f_x = v \cdot f \cdot \sin \alpha, \qquad \text{with } \alpha \in [0; 2\pi] \qquad (7)$$

which is based on inverting the direction of the vertical projection of the cross product $\vec{v} \times \vec{f}$ when the angle between these vectors passes through π. This equation, however, **does not** define α, since the sine function has no inverse in this domain – each sine value in the interval [-1;1] corresponds to two arcs in the interval $[0; 2\pi]$, since sine is positive in the first and second quadrants, and negative in the other two. For this reason, it is necessary to choose, in the domain of the sine function, which *branch* should be used. In other words, it is necessary to establish whether the arcs of the first and fourth quadrants, or of the second and third quadrants are required. Note that we are not talking about the quadrants of the frame of reference *x*O*y*, but that referred to the direction of the velocity; in **fig. 4**, the angle is identified by the thick black line arc.

In order to obtain results around π, we have to choose the branch $\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ for the inverse function:

$$\alpha = \arcsin\left(\frac{v_x f_y - v_y f_x}{v \cdot f}\right), \qquad \text{with } \alpha \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right] \qquad (8)$$

The function $\arcsin(\,)$ of the spreadsheet, however, returns arcs in the range $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$, therefore, the implementation of eq. (8) to correspond to the inverse function of the branch $\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ requires some care.

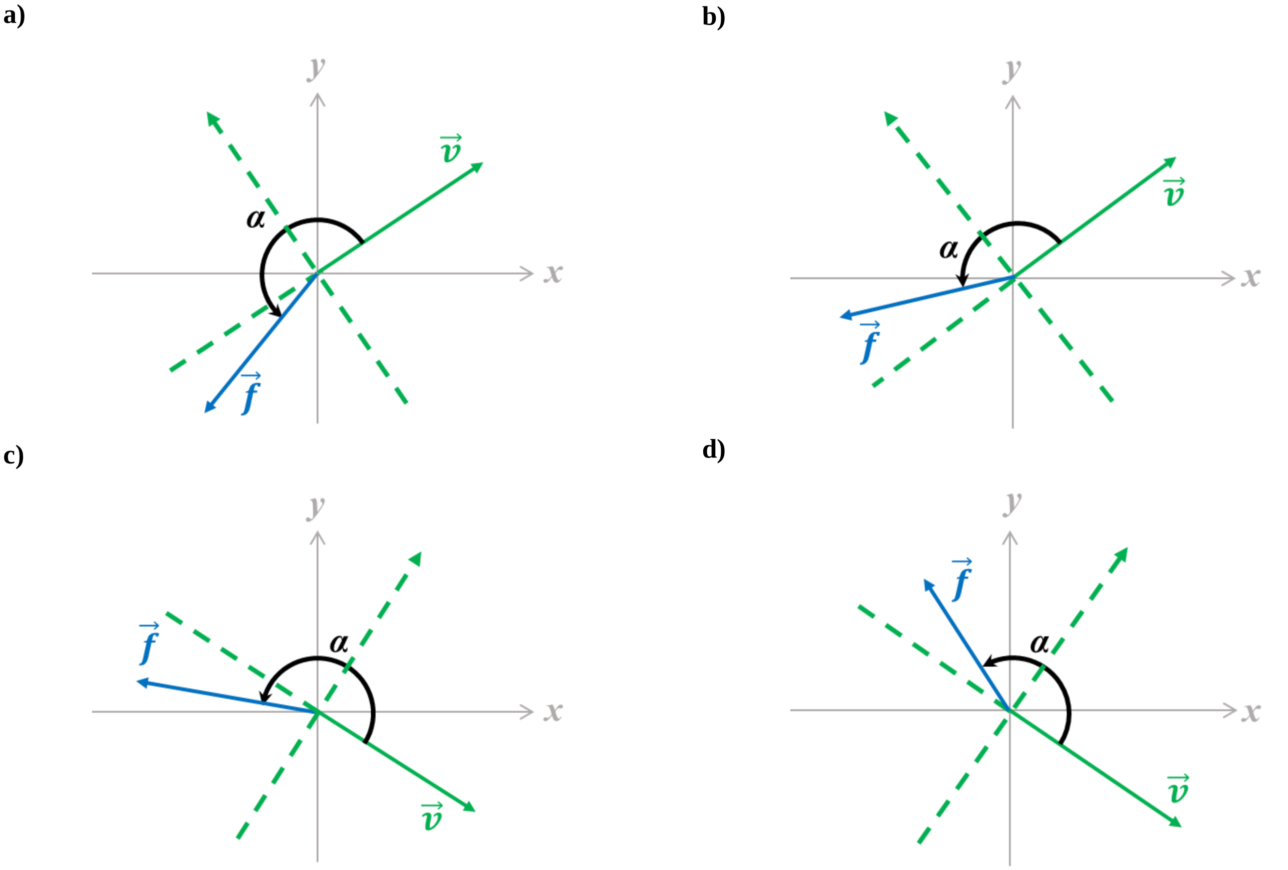

Figure 4. Possible orientations of velocity and resistive force vectors on the plane of motion.
In **a)** and **b)**, the coin has an upward movement and in **c)** and **d)**, downward.
In **a)** and **c)**, $\alpha > \pi$ and the argument of the arcsine function in eq. (8) is negative: $\vec{v} \times \vec{f}$ points into the paper.
In **b)** and **d)**, $\alpha < \pi$ and the argument of the arcsine function in eq. (8) is positive: $\vec{v} \times \vec{f}$ points out of the paper.

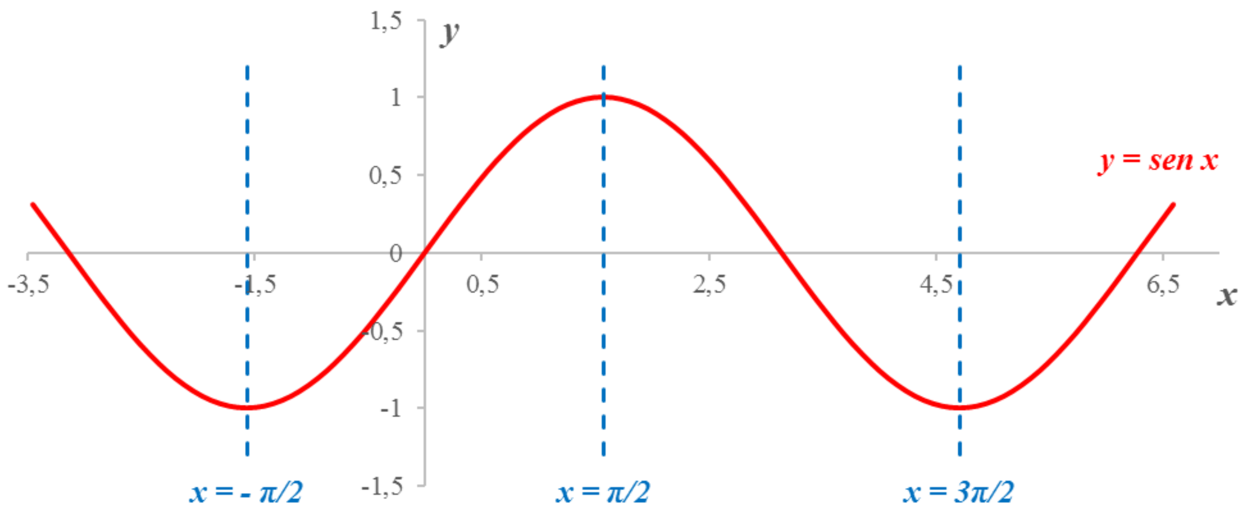

Figure 5. Function graph $y = \sin x$ near $x = {}^{\pi}/_{2}$ rad.

Figure 5 shows the graph of the sine function. Since each invertible branch of the domain of the sine function has an extension of π rad, it is necessary to analyze which angles of the interval $\left[\frac{\pi}{2};\frac{3\pi}{2}\right]$ can be obtained when the values are in the range $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. The intuitive idea of adding π rad to the result is incorrect, because the displacement of the graph branch to $x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ by π rad to the right does not give the branch with $x \in \left[\frac{\pi}{2};\frac{3\pi}{2}\right]$ of the sin function – these branches are *mirrored*. So,

$$\alpha = \pi - \arcsin\left(\frac{v_x f_y - v_y f_x}{v \cdot f}\right) \quad \text{in rad} \tag{9}$$

Written in spreadsheet syntax, eq. (8) corresponds to:

```
=PI()-ASEN((vx*fy-vy*fx)/(v*f))
```

where `vx`, `vy`, `v`, `fx`, `fy` and `f` must refer, respectively, to the cells of the components *x* and *y* and speed modules (`v`) and the resistive force (`f`).

Finally, it is possible to verify that constraining the angle to the range $\left[\frac{\pi}{2};\frac{3\pi}{2}\right]$ is consistent with the experimental data, calculating the dot product $\vec{f} \cdot \vec{v}$, whose sign indicates whether the angle between the vectors is less or greater than $\pi/2$. If this dot product is always negative, the choice of branch for inversion of the sine function will be sufficient to obtain the desired information.

Assessment of uncertainty in the angle between velocity and resistive force

The statistical fluctuation in the measured angle α arises from the uncertainties in the measurements of the components of velocity and of the resistive force, which appears in eq. (9) above, in a particular combination, which we define as:

$$u = \frac{v_x f_y - v_y f_x}{v \cdot f}$$

In this experiment, the relative uncertainty of velocity can be ignored compared to that of the resistive force, so a good approximation can be obtained with:

$$\sigma_u^2 \cong \left(\frac{\partial u}{f_x}\right)^2 \sigma_{f_x}^2 + \left(\frac{\partial u}{f_y}\right)^2 \sigma_{f_y}^2$$

Adopting $\sigma_{f_x}^2 = \sigma_{f_y}^2 = \sigma_f^2$, after evaluating the derivatives and taking some algebraic steps, we arrive at:

$$\sigma_u \simeq \frac{\sigma_f}{f}. \tag{10}$$

Note that on the right side, a relative standard deviation appears, while on the left side, a standard deviation appears, which is the expected result – f is dimensional, while u, not. Returning to eq. (9) and observing the data, which give $\alpha \approx \pi$, we deduce that $u \approx 0$. Using the approximation $\alpha = \sin u \approx u$ valid for $u \approx 0$, we find that the standard deviation of the angle is given by:

$$\sigma_\alpha \simeq \frac{\sigma_f}{f}. \tag{11}$$

Assuming that the intensity of the resistive force is constant (**Fig. 3** shows that this may be true, within the experimental uncertainties), the average value of the resistive force can be used in eq. (11). This leads to the conclusion that the uncertainty in the measurement of the angle α is the same for all time intervals. Finally, notice that formula (11) is also valid for the mean value $\overline{\alpha}$ when the standard deviation of the mean value of the resisting force is used: $\sigma_{\bar{f}_{at}} = 54$ g·cm/s² in the example considered.

Conclusion about the angle between the velocity and the resistive force

Figure 6 presents the experimental values of angle α as a function of time, for the same data set analyzed in **Fig. 3**, where the uncertainties were evaluated by formula (11), using $\sigma_f = 279$ g·cm/s². In **Fig. 6**, it can be seen that 19 of the 27 points are less than one standard deviation from the expected value π rad, and that the mean value of the distribution is $= 3{,}20(7)$ rad – note that $0{,}07$ is the standard deviation of the mean value $\overline{\alpha}$, and not of each data α_i, for a specific moment t_i. Therefore, either by the statistical interpretation of the graph, or by the analysis of the average value, the experimental results are in accordance with the laws of Amontons, which establish that the friction force is opposite to the velocity. So, $\alpha = \pi$ rad will be adopted in the theoretical model. Since the force of kinetic friction will be used in place of the resistive force in the model, the equation of motion for the coin is the one shown in the guide of Part II, eq. (15), which makes it possible to predict the trajectory of the coin on the surface of the inclined plane.

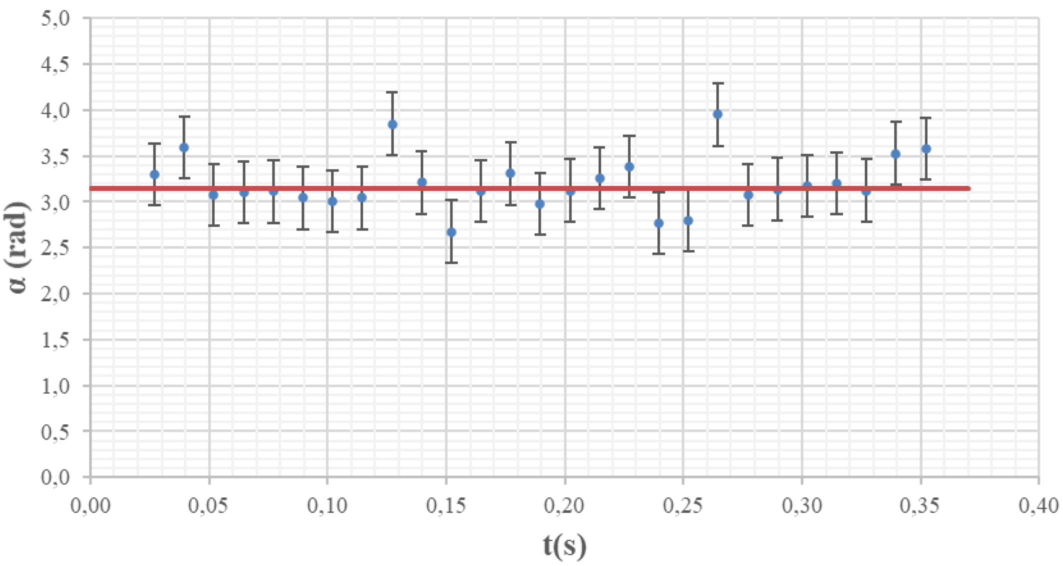

Figura 6. Experimental values of the angle between the velocity and the resistive force on the coin as a function of time, obtained from eq. (9). The uncertainty bars are one standard deviation, see eq. (11). The solid line was drawn at the expected value of π rad.