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Time symmetry breaking in Bose–Einstein condensates

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Abstract

We consider different processes leading to time symmetry breaking in a Bose–Einstein condensate. Our approach provides a global description of time symmetry breaking, based on the equations of a thermal condensate. This includes quenching and expansion of the condensate, the Kibble–Zurek mechanism associated with the creation of vorticity, the dynamical Casimir effect and the formation of time crystals.

Keywords: Bose Einstein condensates, time symmetry breaking, time crystals, dynamical Casimir effect, quenching, Kibble–Zurek mechanism

(Some figures may appear in colour only in the online journal)

1. Introduction

Time symmetry breaking has received considerable attention in recent years, after the suggestion made by Wilczek [1] that spontaneous symmetry breaking could occur in quantum systems at the ground state, leading to the formation of a temporally periodic structure, the so-called 'time crystal'. But it soon became clear that time symmetry breaking cannot occur spontaneously at thermal equilibrium, and can only be observed in systems out of equilibrium [2–6] . This was indeed verified in a quite recent experiment using ion traps [10]. The main purpose of the present work is to examine the problem of time crystallization, in the general context of time symmetry breaking processes in Bose–Einstein condensates (BECs). We will examine the problem taking into account the strong analogies between a time crystal configuration and the dynamical Casimir (DC) effect [4, 7]. There is a considerable difference between ordinary (space) crystals and time crystals, because time is fundamentally different from space. In particular, we cannot travel backwards in time, and such asymmetry is an essential ingredient of temporal optics [11]. This asymmetry is also present in time-varying condensates.

We propose here a global approach to time symmetry breaking in BECs, where temporal effects will be assumed as arbitrary and include, as particular cases, expansion and quenching [8], as well as periodic oscillations of the condensate. In this approach, the Kibble–Zurek (KZ) mechanism for spontaneous vorticity creation [9] will appear as a natural manifestation of the quantum vacuum, which takes place during expansion or quenching. Other phenomena associated with a time-varying condensate have also been considered, such as the DC effect [7, 12]. Whereas time crystals, strongly connected with the DC effect, will appear as a particular aspect of temporally oscillating condensates. They are all consequences of a perturbed bosonic vacuum. In the context of a condensate, we mean a phonon vacuum, or more generally, a Bogoliubov–de Gennes (BdG) vacuum.

The structure of this paper is the following. In section 2 we formulate the basic equations of our model, which is based on the quantum description of the elementary excitations of a condensate. Generally speaking, these excitations are BdG modes, which can be defined in a variety of physical geometries. To be more specific, we examine in section 3 the case of cylindrical condensates, but other geometries such as unbounded or spherical media can equally be considered. In section 4 we derive the dispersion relation for BdG modes, valid for a time varying medium. It will be shown that the mode frequency will not evolve in time according to a simple adiabatic law, but will show a clear non-adiabatic response. This non-adiabatic property scales with the inverse of the characteristic time scale of the medium, and only vanishes for infinitely slow processes.

The equations describing the temporal mode coupling, and determining the creation and annihilation of BdG quanta, will be described in section 5. This section specifies the concept of a sudden quenching, defines the corresponding Bogoliubov transformation, and gives the general solution for mode excitation in a non-stationary medium with an arbitrary temporal behavior. The case of an oscillating condensate will be treated in section 6, where generic time crystals will be defined, and a temporal Bragg condition for resonant scattering or resonant phonon-pair emission will be established. This temporal Bragg's law will indeed specify when the BdG vacuum will become unstable and the dynamical Casimir effect will emerge from a time crystal. This unstable situation has been associated with the phonon analogue of the DC effect [7]. In section 7, the case of a quenched or an expanding condensate is described with generality. We show that, if these processes occur on a fast time scale, generation of vorticity can be observed, which can be associated with the KZ mechanism. A scaling law for vorticity creation is derived, which can be compared with existing experiments. Finally, in section 8, we state some conclusions.

2. Basic equations

We start with the equations describing a BE condensate at finite temperature. We then apply them to the case of a time-varying density, and study the influence of this variation on the structure of the elementary excitations of the medium. In the present work we consider, for simplicity, that the coupling parameter g remains constant. But it should be noticed that the case of a time dependent g could be treated exactly in the same way. This means that the final results can easily be extended to include this situation.

In quite general conditions, the condensate can be described by a bosonic field operator $\hat{\psi} \equiv \hat{\psi}(\mathbf{r}, t)$, which satisfies the Heisenberg equation

$$\mathrm{i}\hbar \frac{\partial \hat{\psi}}{\partial t} = \hat{h}_0 \hat{\psi} + g \hat{\psi}^\dagger \hat{\psi} \hat{\psi}, \quad \hat{h}_0 = -\frac{\hbar^2 \nabla^2}{2m} + V_0(\mathbf{r}) - \mu, \tag{1}$$

where g is the coupling parameter, and $\hat{h}_0$ is the single particle Hamiltonian. Here, μ is the chemical potential, $V_0(\mathbf{r})$ the confining potential and m the single atom mass. We can split the field operator in the usual way, as $\hat{\psi} = \Phi + \hat{\eta}$, where $\left\langle \hat{\psi} \right\rangle = \Phi$ is the mean field, and $\langle \hat{\eta} \rangle = 0$. This description is valid if we assume the condensate as a coherent superposition of states with different values of the total number of atoms N [13]. In this case, the averaging procedure is valid, as long as the fraction of non-condensed atoms remains small. Replacing this in equation (1) and using the mean field approximation, we obtain an evolution equation for the mean field Φ, as

$$\mathrm{i}\hbar \frac{\partial \Phi}{\partial t} = \left[\hat{h}_0 + g(n_c + 2n_{\mathrm{th}}) \right] \Phi + g n_a \Phi^*, \tag{2}$$

and another equation for the thermal operator $\hat{\eta}$,

$$\mathrm{i}\hbar \frac{\partial \hat{\eta}}{\partial t} = \left[\hat{h}_0 + 2g(n_c + n_{\mathrm{th}}) \right] \hat{\eta} + g \left(\Phi^2 + n_a \right) \hat{\eta}^\dagger. \tag{3}$$

In these equations we have used the condensate density $n_c = |\Phi|^2$, the thermal density $n_{\mathrm{th}} = \left\langle \hat{\eta}^\dagger \hat{\eta} \right\rangle$, and the anomalous density $n_a = \langle \hat{\eta}\hat{\eta} \rangle$. Using the Popov approximation, $n_a \simeq 0$, and defining the total atomic density as $n = n_c + n_{\mathrm{th}}$, we can rewrite equation (3) more simply as

$$\mathrm{i}\hbar \frac{\partial \hat{\eta}}{\partial t} = \left[\hat{h}_0 + 2gn \right] \hat{\eta} + g\Phi^2 \hat{\eta}^\dagger. \tag{4}$$

This will be the starting point of our model. The Popov approximation is valid for a finite temperature condensate, under small amplitude mechanical disturbances [14, 15] such that $|n_a| \ll |\Phi^2|$, and will eventually break down for extreme quenching.

3. Elementary excitations

Our main purpose is to understand the evolution of the thermal operator $\hat{\eta}$, as described by the above evolution equation, in the case of a non-stationary condensed background. For this purpose we define $\hat{\eta}$ as a superposition of elementary excitations, parametrized by the index k, to be specified later. We write $\hat{\eta} = \sum_k \hat{b}_k(\mathbf{r}, t)$, where the mode operators satisfy the usual bosonic commutation relations $\left[\hat{b}_k^\dagger, \hat{b}_{k'} \right] = \delta(k - k')$, and $\left[\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger \right] = \left[\hat{b}_k, \hat{b}_{k'} \right] = 0$. We focus on excitations propagating along the z-axis, in a cylindrical geometry. Other geometric configurations, such as spherical or unbounded condensates can easily be described in a similar way. In the assumed geometry, we can use an explicit mode structure, of the form

$$\hat{b}_k(\mathbf{r}, t) = \hat{b}_k(\mathbf{r}_\perp, t) \exp(\mathrm{i}k_z z), \quad \hat{b}_k(\mathbf{r}_\perp, t) = \sum_{l,p} \hat{b}_{klp}(t) F_{lp}(\mathbf{r}_\perp). \tag{5}$$

The transverse mode structure is defined here with generality, as a superposition of orthogonal Bessel modes $F_{lp}(\mathbf{r}_\perp) = c_{lp} J_l(\alpha_{lp} r/a) \exp(\mathrm{i}l\theta)$, where we have used polar coordinates $\mathbf{r}_\perp \equiv (r, \theta)$. Here, the quantities α_{lp} are the successive zeros of the Bessel function J_l, and a is the condensate radius. We also choose the normalization constants as $c_{lp} = a\sqrt{\pi} J_{l+1}(\alpha_{lp})$. This choice of solution guarantees that the excitation amplitudes are zero at the boundary, $r = a$, but allow for the existence of an arbitrary number of nodes inside the medium. The internal products between different radial functions satisfy

$$\langle F_{lp}|\, F_{l'p'}\rangle \equiv \int_0^a r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta F_{lp}^*(\mathbf{r}_\perp) F_{l'p'}(\mathbf{r}_\perp) = \delta_{ll'}\delta_{pp'}. \tag{6}$$

It can easily be shown [18] that the normal modes defined by equation (5) satisfy the Helmholtz equation

$$\left(\nabla^2 + k^2\right) \hat{b}_k(\mathbf{r}, t) = 0, \quad k^2 = k_\perp^2 + k_z^2, \tag{7}$$

where k_z was the chosen value for the axial wavenumber, and the perpendicular wavenumber which emerges from the Bessel function solutions is defined as $k_\perp = \alpha_{lp}/a$. Replacing equations (5) and (7) in the evolution equation for the condensate operator (4), we obtain

$$\mathrm{i}\hbar \frac{\partial \hat{b}_k}{\partial t} = H_k \hat{b}_k + g\Phi^2 \hat{b}_{-k}^\dagger, \quad -\mathrm{i}\hbar \frac{\partial \hat{b}_{-k}^\dagger}{\partial t} = H_k \hat{b}_{-k}^\dagger + g\Phi^{*2} \hat{b}_k, \tag{8}$$

with

$$H_k = \frac{\hbar^2 k^2}{2m} + V_0(\mathbf{r}) + 2gn - \mu. \tag{9}$$

If we now multiply these equation by $F_{l'p'}^*$, integrate in the perpendicular coordinate $\mathbf{r}_\perp$ and use the orthogonality condition (6), we obtain a new system of coupled mode equations, of the form

$$\mathrm{i}\hbar \frac{\partial \hat{b}_q}{\partial t} = h_q \hat{b}_q + S_q \hat{b}_{-q}^\dagger, \quad -\mathrm{i}\hbar \frac{\partial \hat{b}_{-q}^\dagger}{\partial t} = h_q \hat{b}_{-q}^\dagger + S_q^* \hat{b}_q. \tag{10}$$

Here, we have used a simplified mode notation, such that $q \equiv (k, l, p)$, and $-q \equiv (-k, -l, p)$, and defined the new quantities

$$h_q \equiv \langle H_q \rangle = \langle F_{lp} \left| H_k \right| F_{lp} \rangle, \quad S_q = g \langle F_{lp} \left| \Phi^2 \right| F_{lp} \rangle. \tag{11}$$

Notice that, by assuming a specific radial structure of the modes, satisfying the Helmholtz equation (7) for a given wavenumber k_z, we were able to reduce the mode equations to this simple form. The advantage of the new equations (10), is that they no longer depend on the radial coordinate $\mathbf{r}_\perp$, and for this reason they represent the global properties of the elementary excitations in the cylindrical condensate (see [19]).

4. BdG modes

The main properties of the elementary modes in steady-state condensates are well understood, but not under time symmetry breaking. The interest of the above operator equations is that they can be used to derive the mode dispersion relations, in the case of a time-dependent condensate. It will be seen that time dependence introduces a new contribution to mode dispersion, which is clearly non-adiabatic. Validity conditions for the adiabatic approximation can also be established. In order to describe these effects, we take the time derivative of equation (10), and get

$$\frac{\partial^2 \hat{b}_q}{\partial t^2} + \left[\omega_{q0}^2 + \frac{\mathrm{i}}{\hbar}\frac{\partial h_q}{\partial t}\right] \hat{b}_q + \frac{\mathrm{i}}{\hbar}\frac{\partial S_q}{\partial t} \hat{b}_{-q}^\dagger = 0, \quad \frac{\partial^2 \hat{b}_{-q}^\dagger}{\partial t^2} + \left[\omega_{q0}^2 - \frac{\mathrm{i}}{\hbar}\frac{\partial h_q}{\partial t}\right] \hat{b}_{-q}^\dagger - \frac{\mathrm{i}}{\hbar}\frac{\partial S_q^*}{\partial t} \hat{b}_q = 0, \tag{12}$$

Here, we have defined a time-dependent frequency $\omega_{q0} \equiv \omega_{q0}(t)$, as

$$\omega_{q0}^2 = \frac{1}{\hbar^2}\left[h_q^2 - |S_q|^2\right]. \tag{13}$$

At this point, it is useful to consider a Thomas–Fermi density profile, such that $gn = \mu - V_0(r)$. This density profile stays valid for a condensate with a large number of atoms, or a large coupling parameter g, such that the kinetic energy can be neglected as compared with gn. The breakdown of this approximation near the boundary, at $r = a$, will not significantly change the radially averaged quantities used in the present model. The quantity h_q will reduce to

$$h_q = \frac{\hbar^2 k^2}{2m} + gn_q, \quad n_q = \langle F_{lp}|\, n\, |F_{lp}\rangle. \tag{14}$$

On the other hand, noting that $\Phi^2 = n_c \exp(-\mathrm{i}\varphi_0)$, where φ_0 is a constant phase, we can also write

$$S_q = gn_{cq}\exp(-\mathrm{i}\varphi_0), \quad n_{cq} = \langle F_{lp}|\, n_c\, |F_{lp}\rangle. \tag{15}$$

This means that, for a constant g, the time derivatives of h_q and S_q in equation (12) are simply given by

$$\frac{\partial h_q}{\partial t} = g\frac{\partial n_q}{\partial t}, \quad \frac{\partial S_q}{\partial t} = g\frac{\partial n_{cq}}{\partial t}\mathrm{e}^{-\mathrm{i}\varphi_0}. \tag{16}$$

Using this in equation (12), we can rewrite them as

$$\frac{\partial^2 \hat{b}_q}{\partial t^2} + \omega_q^2 \hat{b}_q + \frac{\mathrm{i}g}{\hbar}\frac{\partial n_{cq}}{\partial t}\mathrm{e}^{-\mathrm{i}\varphi_0}\hat{b}_{-q}^\dagger = 0, \quad \frac{\partial^2 \hat{b}_{-q}^\dagger}{\partial t^2} + \omega_q^2 \hat{b}_{-q}^\dagger - \frac{\mathrm{i}g}{\hbar}\frac{\partial n_{cq}^*}{\partial t}\mathrm{e}^{\mathrm{i}\varphi_0}\hat{b}_q = 0, \tag{17}$$

where we have defined a *modified mode frequency*, $\omega_q \equiv \omega_q(t)$, such that

$$\omega_q^2(t) = \omega_{q0}^2(t) \pm \frac{g}{\hbar}\frac{\partial n_q}{\partial t}. \tag{18}$$

In this expression, the plus sign $(+)$ refers to the first of equation (17), and the minus sign $(-)$ to the second equation. This sign difference is indeed irrelevant. This can be seen, using the following argument. If we consider the Fourier spectrum of the density oscillations, as

$$n_q(t) = \int n_q(\Omega)\exp(\pm\mathrm{i}\Omega t)\frac{\mathrm{d}\Omega}{2\pi}, \tag{19}$$

we should use, for consistency, the minus sign of this expression in the first of equation (17), and the plus sign in the second one. Taking this into account we can transform equation (18) into

$$\omega_q^2(t) = \omega_{q0}^2(t) + \frac{g}{\hbar}\int \Omega n_q(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\frac{\mathrm{d}\Omega}{2\pi}. \tag{20}$$

We can now define the average frequency $\bar{\Omega}$, characterizing the characteristic time scale of the density variations, as

$$\bar{\Omega} = \frac{1}{n_q(t)}\int \Omega n_q(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\frac{\mathrm{d}\Omega}{2\pi}, \tag{21}$$

and defining the Bogoliubov speed for the specific q-mode, as $c_{sq}^2 = gn_q(t)/m$, we can write equation (20) as

$$\omega_q^2(t) = \omega_{q0}^2(t) + \frac{m}{\hbar}\bar{\Omega}c_{sq}^2. \tag{22}$$

This is the dispersion relation for the BdG q-mode in a time-varying condensate. It states one of the main properties of time symmetry breaking, the mode frequency shift. We can see that the mode frequency changes, not just due to the variation of the background density (this would explain the first term), but is also dependent on the typical time scale of the evolving density spectrum (second term). This frequency shift is clearly non-adiabatic. The adiabatic regime should only be valid for very slow processes, such that

$$\bar{\Omega} \ll \frac{\hbar}{m} \frac{\omega_{q0}^2}{c_{sq}^2} \tag{23}$$

In this limit, the dispersion relation (22) would then reduce to

$$\omega_q^2(t) \simeq \omega_{q0}^2(t) = k^2 c_{sq}^2(t) + \frac{\hbar^2 k^4}{4m^2}. \tag{24}$$

The existence of a time dependent mode frequency is somewhat counter-intuitive, but it has been considered for a long time in theory and experiments (see, for instance, the recent work by Yang [20] and references therein).

5. Boson excitation

Apart from the frequency shift just described, the time symmetry breaking process also leads to pair emission from a boson vacuum. A similar process is known in quantum optics, and was analyzed for photons in time-varying optical media in [11]. We start from the mode coupled equation (17), and introduce a new pair of mode operators $(\hat{a}_q, \hat{a}_q^\dagger)$, as

$$\hat{b}_q = \hat{a}_q \exp\left[-\mathrm{i}\varphi_q(t) - \mathrm{i}\varphi_0\right], \quad \hat{b}_{-q}^\dagger = \hat{a}_{-q}^\dagger \exp\left[+\mathrm{i}\varphi_q(t) + \mathrm{i}\varphi_0\right], \tag{25}$$

with the phase function defined as

$$\varphi_q(t) = \int^t \omega_q(t')\mathrm{d}t'. \tag{26}$$

We also assume that the new operators are nearly constant on the time scale $1/\omega_q$, which is equivalent to state that

$$\left| \frac{\partial}{\partial t} \left(\hat{a}_q, \hat{a}_{-q}^\dagger\right) \right| \ll \left| \omega_q \left(\hat{a}_q, \hat{a}_{-q}^\dagger\right) \right|. \tag{27}$$

Replacing this in equation (17), we can reduce them to the simple standard form

$$\frac{\partial \hat{a}_q}{\partial t} = \nu_q(t) \hat{a}_{-q}^\dagger, \quad \frac{\partial \hat{a}_{-q}^\dagger}{\partial t} = \nu_q(t)^* \hat{a}_q, \tag{28}$$

where we have introduced the quantity

$$\nu_q(t) = \frac{g}{2\hbar\omega_q} \frac{\partial n_{cq}}{\partial t} e^{2i\varphi_q}. \tag{29}$$

These are the two basic equations associated with time symmetry breaking. They show that the two modes q and $-q$, propagating in opposite directions along the z-axis, and possessing orbital angular momenta of opposite signs, as determined by the poloidal quantum numbers l and $-l$, are coupled with each other due to the temporal variations of the medium. A similar process occurs in optics, as noted above. Integration of these equations leads to the following solution

$$\hat{a}_q(t) = \cosh[r_q(t)]\hat{a}_q(0) + \sinh[r_q(t)]\hat{a}^\dagger_{-q}(0),$$
$$\hat{a}^\dagger_{-q}(t) = \cosh[r_q(t)]\hat{a}^\dagger_{-q}(0) + \sinh[r_q(t)]\hat{a}_q(0)\,, \tag{30}$$

where we have defined the *squeezing parameter*

$$r_q(t) = \int^t |\nu_q(t')|\,\mathrm{d}t'. \tag{31}$$

As an example, it is interesting to consider the particular case of a 'sudden quenching', often discussed in the literature [16], where the condensate density suddenly jumps from an initial value n_{q1} to a final value n_{q2}. This case can be described by the simple law

$$n_{cq}(t) = n_1 + \Delta n_q H(t), \tag{32}$$

where $\Delta n_q = n_2 - n_1$ defines the density jump, and the Heaviside function $H(t)$ determines the quench duration at $t = 0$. In this case, the squeezing parameter (31) reduces to $r_q(t) = (g/2\hbar)\Delta n_q$ and equation (30) take the form of a standard Bogoliubov transformation, relating the old mode operators $\hat{a}_{q1} = \hat{a}_q(t < 0)$, and $\hat{a}^\dagger_{-q1} = \hat{a}^\dagger_{-q}(t < 0)$, to the new ones $\hat{a}_{q2} = \hat{a}_q(t \geqslant 0)$, and $\hat{a}^\dagger_{-q2} = \hat{a}^\dagger_{-q}(t \geqslant 0)$. We obtain

$$\hat{a}_{q2} = A\hat{a}_{q1} + Ba^\dagger_{-q1}, \quad \hat{a}^\dagger_{-q2} = A\hat{a}^\dagger_{-q1} + Ba_{q1}, \tag{33}$$

with the coefficients A and B defined as

$$A = \cosh\left[\frac{g}{2\hbar}\Delta n_q\right], \quad B = \sinh\left[\frac{g}{2\hbar}\Delta n_q\right]. \tag{34}$$

This satisfies the usual bosonic condition $A^2 - B^2 = 1$, which is characteristic of an hyperbolic transformation.

The above general solutions (30) describe the emission of boson-pairs from vacuum, due to the temporal changes of the medium. This general property is well understood in quantum optics, where it concerns the electromagnetic photon vacuum: here we are concerned with the BdG phonon vacuum associated with the elementary excitations of the Bose–Einstein condensate. These two different vacuum fields are distinct in what concerns the spin states of the elementary quanta, but their bosonic properties are very similar. We can define the quantum number operator for a given BdG mode, in the usual way, as $N_q(t) = \hat{a}^\dagger_q(t)\hat{a}_q(t)$. Using equation (30), we can then determine the number of quantum pairs associated with the modes q and $-q$, due to the temporal variation of the condensate. The result is

$$\langle N_q(t)\rangle = \sinh^2[r_q(t)]\,. \tag{35}$$

It can be seen, from the definition of the parameter $r_q(t)$, given in equations (29)–(31), that the number of emitted quantum pairs is only significant if the temporal variations of the condensate density $n_{cq}(t)$ occurs on a time scale of the order of $1/2\omega_q$. We can now explore the above results in a couple of typical situations where such temporal variations become possible. One is the so-called 'time-crystal', where the density of the condensed atoms oscillates periodically in time. The other corresponds to an irreversible variation of the condensate density, which can be associated to two opposite processes: quenching and expansion.

6. Time crystals

Let us first focus on an oscillating condensate, which would lead to the formation of a time crystal. This temporal structure occurs in the presence of a periodic perturbation of the condensate density or, alternatively, of the coupling parameter g. These two cases are nearly identical. We can define a time crystal, with generality, by assuming a temporal evolution of the form [4]

$$n_{cq}(t) = n_{cq}(0)\left[1 + \epsilon G(t) f(t)\right], \tag{36}$$

where $\epsilon = \delta n_q / n_{cq}(0)$ is the density modulation amplitude, $f(t) = f(t + \tau_c)$ is a periodic function with period τ_c, and $G(t)$ is a slowly varying function which determines the shape of the crystal envelope and is non-zero for a duration $T \gg \tau_c$. The quantity T defines the size of the time crystal. We can use a Fourier series, and write

$$f(t) = \sum_{-\infty}^{\infty} f_n \exp(\mathrm{i} n\omega_c t), \tag{37}$$

with $\omega_c = 2\pi/\tau_c$, and arbitrary coefficients f_n. Replacing this in equation (29), we obtain

$$\nu_q(t) = \sum_{-\infty}^{\infty} \nu_{qn}(t) \exp\left[\mathrm{i} n\omega_c t - 2\mathrm{i}\varphi_q(t)\right], \tag{38}$$

with slowly varying coefficients determined by

$$\nu_{qn}(t) = \mathrm{i} n \epsilon f_n \frac{g}{2\hbar} \frac{\omega_c}{\omega_q} G(t). \tag{39}$$

To be specific, let us focus on a simple sinusoidal crystal, such that $f(t) = \cos(\omega_c t)$, with a sharp envelope function $G(t) = H(t) - H(t - T)$, where $H(t)$ is Heaviside. In this case, we can write equation (22) in form

$$\omega_q^2(t) = \omega_{q0}^2 + k^2 c_{sq}^2(0)\epsilon \cos(\omega_c t). \tag{40}$$

Here, the quantity ω_{q0} is a constant, given by

$$\omega_{q0}^2 = k^2 c_{sq}^2(0) + \frac{\hbar^2 k^4}{4m^2} + g n_{cq}\omega_c. \tag{41}$$

The adiabatic limit would correspond here to $\omega_c \to 0$. Using equation (26), we can write the phase function as

$$\varphi_q(t) = \omega_{q0} t + \frac{\kappa}{2} \sin(\omega_c t), \tag{42}$$

with

$$\kappa = \epsilon \frac{k^2 c_{sq}^2(0)}{\omega_c \omega_p} \sim \epsilon \frac{\omega_{q0}}{\omega_c}. \tag{43}$$

Replacing this in equation (29), and developing the exponential in Bessel functions, we obtain

$$\nu_q(t) = -\frac{g\epsilon}{2\hbar} \frac{\omega_c}{\omega_q} \sin(\omega_c t) \sum_{\nu=-\infty}^{\infty} J_\nu(\kappa) \exp\left[\mathrm{i}(2\omega_{q0} + \nu\omega_c)t\right]. \tag{44}$$

Resonant contributions to the squeezing parameter $r_q(t)$ will arrive from the constant terms in this expansion. They occur for a given value of the integer ν, such that

$$\omega_{q0} = (\nu \pm 1)\frac{1}{2}\omega_c. \tag{45}$$

This is equivalent to choosing ν as the integer part of $(2\omega_{q0}/\omega_c) \mp 1$. It defines the *Bragg condition* for resonant temporal diffraction of the incoming BdG mode onto the time crystal.

Replacing this resonance condition in equation (35) we obtain, for the number of modes excited from vacuum by such a temporal structure

$$\langle N_q(t)\rangle = \sinh^2(\bar{\nu}_q T), \quad \bar{\nu}_q = \frac{g\epsilon}{4\hbar}\frac{\omega_c}{\omega_q}. \tag{46}$$

This result shows that, for short crystals, we get a quadratic growth for the number of modes, as $\langle N_q(t)\rangle \simeq \bar{\nu}_q^2 T^2$, proportional to the square of the crystal size. In contrast, for large crystals, we get an exponential growth, given by

$$\langle N_q(t)\rangle \simeq \frac{1}{4}\exp\left(2\bar{\nu}_q T\right). \tag{47}$$

This number can be significant for $T \gg 1/2\bar{\nu}_q$. Such a vacuum instability associated with long time crystals can be seen as the phonon analogue of the 'dynamical Casimir effect' [7]. Similar time crystals can also be excited in non-condensed laser-cooled gas, as proposed elsewhere [4]. This also has formal similarities with the parametric instabilities of the mean field as considered [17], although we consider here the quantum elementary excitations and not the mean field quantities. Of course, the exponential growth is only valid in the linear stages of the instability, and it will eventually saturate, due to condensate depletion, as briefly discussed in the conclusions.

7. Quenching

The previous section focused on periodic media, and in order to complete our analysis we need to consider now the irreversible processes, such as quenching (and expansion) of the condensate. If these processes take place on a time-scale τ, we can describe them using the following law

$$n_{cq}(t) = n_{cq}(0) + \frac{\Delta n_q}{2}\left[1 + \tanh(t/\tau)\right], \tag{48}$$

The quenching process is illustrated in figure 1, where the frequency variation of a given BdG q-mode is represented as a function of time. This is distinct from what should be expected from a simple adiabatic process, as illustrated in the figure. For quenching, we use $\Delta n_q > 0$, and for very strong quenching we can eventually assume a large density variation, $\Delta n_q \gg n_{cq}(0)$. In contrast, for an expansion process, we always have $\Delta n_q < 0$, and $|\Delta n_q| \simeq n_{cq}(0)$ such that the density with eventuality tends asymptotically to zero for $t \to \infty$. From the discussion in section 5, we can see that a significant excitation of a given BdG q-mode due to quenching or expansion only occurs for fast processes, such that $\tau \leqslant 1/2\omega_q$. In order to understand this mode excitation in more detail, let us replace equations (48) in (29)–(31). In this case, we have

$$f(t) \equiv \frac{\partial n_{cq}}{\partial t} = \frac{\Delta n_q}{2\tau}\mathrm{sech}^2(t/\tau). \tag{49}$$

We then get

$$r_q(t) = \int^t |\nu_q(t')|\,\mathrm{d}t' = \frac{g\Delta n_q}{4\hbar\tau}\int^t \left|\frac{1}{\omega_q(t')}\mathrm{sech}^2(t'/\tau)\mathrm{e}^{2\mathrm{i}\varphi_q(t')}\right|\mathrm{d}t'. \tag{50}$$

It is useful to take the Fourier transform of the function $f(t) = \int f(\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega/2\pi$, defined in equation (49). The result is

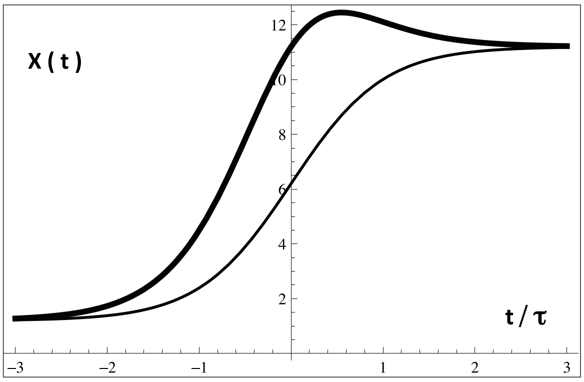

Figure 1. Normalized mode frequency square $X(t)$ (bold curve), with $X(t) = \omega_q(t)^2/k^2c_{sq}^2(t)$ for a quenching process. We have used, for illustration, $\Delta n = 10$ and $k^2 = mn(0)/\hbar$. The adiabatic value for the mode frequency is also shown (thin curve).

$$f(\omega) = \frac{\Delta n_q}{2\sqrt{2}}\sqrt{\pi}\tau \,\mathrm{csch}\left(\frac{\pi}{2}\omega\tau\right). \tag{51}$$

Replacing this in equation (50), we obtain

$$r_q(t) = \frac{g\Delta n_q}{4\hbar\omega_{q0}}\sqrt{\frac{\pi}{2}}\tau \int^t \mathrm{d}t' \int \frac{\mathrm{d}\omega}{2\pi} \,\mathrm{csch}\left(\frac{\pi}{2}\omega\tau\right) \exp\left\{-\mathrm{i}[\omega t' - 2\varphi_q(t')]\right\}. \tag{52}$$

It is obvious that the main contribution to the value of $r_q(t)$ is given by the frequency component $\omega = 2\omega_{q0}$. Extending the time integration to infinity, we can then define the asymptotic value for the squeezing parameter $r_q(\tau) \equiv r_q(t \to \infty)$, as

$$r_q(\tau) = \sqrt{\frac{\pi}{2}}\frac{g\tau\Delta n_q}{2\hbar\omega_{q0}} \,\mathrm{csch}\left(\pi\omega_{q0}\tau\right). \tag{53}$$

It is well known that the hyperbolic function in this expression tends to zero with $\tau\omega_{q0}$. This means that, for a slow quench duration scale such that $\tau \gg 1/\omega_{q0}$, the emission of BdG phonon pairs will be negligible. In contrast, of a short quench duration such that $\tau \ll 1/\omega_{q0}$, we can use the approximation $\mathrm{csch}(x) \simeq 1/x$, which is valid for $x \ll 1$, and obtain

$$r_q(\tau) = r_0 \equiv \frac{\Delta n}{2\sqrt{2\pi}}\frac{gT_q}{\hbar}, \tag{54}$$

with $T_q = 1/\omega_{q0}$. It is indeed independent of the quench duration τ, as long as $\tau \ll 1/\omega_{q0}$. For $x \geqslant 1$, we could use the series expansion [21] $\mathrm{csch}(x) \simeq 1/x - x/6 + 7x^3/360$. For $x \leqslant 2$ the first two terms could be used as a good approximation. We then get, for the asymptotic value

$$r_q(\tau) = r_0\left(1 - \frac{\pi}{6}\frac{\tau^2}{T_q^2}\right). \tag{55}$$

Here, the first term is the quantity r_0 defined above, and the second term is the first correction associated with a finite quench duration. It is then clear that, by increasing the duration of the quenching or expansion process we can reduce the squeezing parameter and, as a consequence,

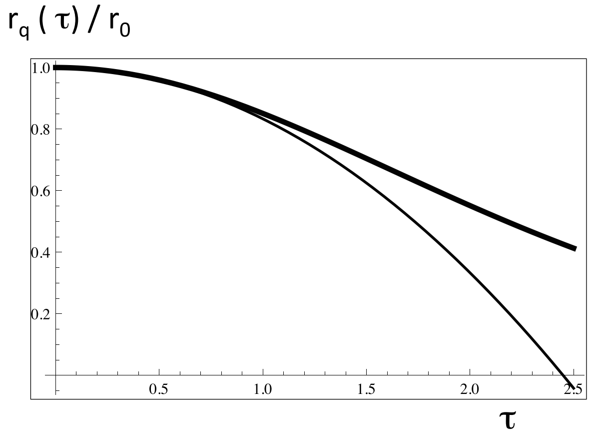

Figure 2. Asymptotic value of the normalized squeezing parameter $r_q(\tau)/r_0$, as a function of the quench duration τ (bold curve). The approximate expression of equation (55) is also shown for comparison.

the number of excited BdG quantum pairs. The dependence of the squeezing parameter with the process duration is represented in figure 2. At this point, we should note that each BdG mode, defined by the quantum numbers $q \equiv (k, l, p)$, carries a finite amount of vorticity, with a z-component of the angular momentum given by $\hbar l$. We are then led to the conclusion that quenching or expansion will introduce vorticity into the system. This is, in essence, the KZ mechanism. We can then define the 'average vorticity', or the average winding number, as

$$\langle L(t) \rangle = \sum_q |l| \, \langle N_q(t) \rangle \,, \tag{56}$$

This quantity should be evaluated for times of the order (or larger than) τ. Noting that $\sinh(x) \simeq x$, for $x \leqslant 1$, we can easily estimate from the above expressions, the amount of vorticity created by quenching or expansion. We can then easily establish the following power law for the KZ mechanism

$$\langle L(t > \tau) \rangle \, \propto \, \tau^{\alpha}, \quad \alpha = 2 - \beta. \tag{57}$$

Looking back at equation (55) we can easily conclude that the parameter β is of the order but slightly larger than 2, which means that α will be a small and negative number. This is in qualitative agreement with recent expansion experiments by [22], where it was found that $\beta = 2.2$. We clearly see that the KZ mechanism is naturally included in the time symmetry breaking process associated with quenching or expansion, and that the corresponding power laws are consistent with the experiments. Another important point is related with the conservation of total angular momentum. This means that, for cylindrically symmetric temporal perturbations, pairs of modes with equal amplitudes but opposite winding numbers, l and $-l$, would be excited. They would equally contribute to the average vorticity, as defined by equation (56). Only for non-axially symmetric temporal perturbations, as imposed by external sources, would the total angular momentum of the condensate change.

8. Conclusions

In this paper, we have described time symmetry breaking in Bose–Einstein condensates. Using a quantum description of BdG modes in a time-varying condensate, we have shown that counter-propagating modes become coupled due to the temporal variations of the medium. We have derived quantum operator equations describing the coupled mode oscillations, and established the corresponding solutions. They show that, due to temporal variations, pairs of BdG quantum modes can be excited from vacuum. In this context, vacuum means absence of thermal fluctuations, and corresponds to a condensate at zero temperature. Thermal effects will then be introduced by the time symmetry breaking process. But we cannot define a temperature, because the fluctuation spectrum will depend on the actual evolution of the medium, and will be quite different from a thermal spectrum.

Two relevant cases were considered in detail. One corresponds to an oscillating condensate, where time crystals can be formed. A temporal Bragg condition was derived, corresponding to resonant scattering. This vacuum instability can lead to an exponential growth of BdG modes, which is a characteristic feature of the dynamical Casimir effect. The other corresponds to expansion or quenching, and leads to the excitation of finite vorticity. Such processes display the characteristic features of the Kibble–Zurek mechanism. We can then conclude that a variety of time symmetry breaking processes can be described by the same quantum mode equations. It should be noticed that, both the KZ mechanism and the dynamical Casimir effect have already been studied experimentally in condensates [7, 22]. Qualitative agreement with our model has already been noticed. We hope that the present work will provide a consistent theoretical framework for the planning and understanding of future and more detailed experiments.

In the present work we have used some simplifying assumptions. The most relevant one is related with the condensate depletion, which was ignored for simplicity. It is obvious that, due to the possible excitation of BdG modes, a significant depletion will eventually occur. This is particularly true in the case of time crystals, if the dynamical Casimir instability is attained, because of the exponential growth of the resonate modes. Depletion will then contribute to the instability saturation, and can be described by a perturbative approach or, alternatively, by a numerical integration of the mode equations. This is a natural extension of the present model, which will be examined in a future publication.

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